

# THE $M_2$ -RANK OF PARTITIONS WITHOUT REPEATED ODD PARTS AS A MOCK MODULAR FORM

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**ABSTRACT.** While it is known that the  $M_2$ -rank of partitions without repeated odd parts is the so-called holomorphic part of a certain harmonic Maass form, much more can be done with this fact. We greatly improve the standing of this function as a harmonic Maass form, in particular we show the related harmonic Maass form transforms like the generating function for partitions without repeated odd parts (which is a modular form). We then use these improvements to determine formulas for the rank differences modulo 7. Additionally we give identities and formulas that allow one to determine formulas for the rank differences modulo  $c$ , for any  $c > 2$ .

## 1. INTRODUCTION

To begin we recall that a partition of an integer is a non-increasing sequence of positive integers that sum to  $n$ . For example the partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, and 1 + 1 + 1 + 1 + 1. It is standard to let  $p(n)$  denote the number of partitions of  $n$ . From our example we see that  $p(5) = 7$ . Partitions have a rich history in number theory and combinatorics, going as far back as Euler. Today much emphasis is put on Ramanujan's work with partitions and his related work with  $q$ -series.

One point of interest are the congruences of Ramanujan,

$$\begin{aligned} p(5n + 4) &\equiv 0 \pmod{5}, \\ p(7n + 5) &\equiv 0 \pmod{7}, \\ p(11n + 6) &\equiv 0 \pmod{11}. \end{aligned}$$

Dyson in [8] proposed the following combinatorial explanation of the first two congruences. Given a partition of  $n$ , we define the rank of the partition to be the largest part minus the number of parts. If we group the partitions of  $5n + 4$  according to the modulo 5 value of their rank, then it turns out we have five sets of equal size. If we group the partitions of  $7n + 5$  according to the modulo 7 value of their rank, then it turns out we have seven sets of equal size. These two statements were later proved by Atkin and Swinnerton-Dyer in [2].

Another point of interest are mock theta functions, which Ramanujan introduced in his famous last letter to Hardy. Mock theta functions are the topic of Watson's final address as president of the London Mathematical Society [21]. We will save the more technical discussion for the next section, but we give two of Ramanujan's examples:

$$\begin{aligned} f(q) &= 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots, \\ \phi(q) &= 1 + \frac{q}{(1+q^2)} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots \end{aligned}$$

One of Ramanujan's mock theta conjectures (which was proved by Watson in [21]) is that  $2\phi(-q) - f(q) = \vartheta_4(0, q) \prod_{n=1}^{\infty} (1 + q^n)^{-1}$ , where  $\vartheta_4(0, q)$  is a Jacobi theta function.

These two areas are actually strongly connected. In particular, if we let  $N(m, n)$  denote the number of partitions of  $n$  with rank  $m$ , we have the generating function given by

$$R(\zeta; q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) \zeta^m q^n.$$

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Of the seven functions commonly referred to as the third order mock theta functions, all seven can be expressed in terms of  $R(\zeta; q)$  by replacing  $q$  by a power of  $q$ , letting  $\zeta$  be a power of  $q$  times a root of unity, and possibly adjusting by a constant and a power of  $q$ . It is worth pointing out that it was Watson [21] who observed the third order functions could be written in terms of the rank, and he did so nearly a decade before the rank function was formally defined. The fifth and seventh order mock theta functions can also be obtained in such a manner, if one also allows adding on certain infinite products. Explicit versions of these statements can be found in section 5 of [12]. For this reason  $R(\zeta; q)$  is called a universal mock theta function. With this we see a strong understanding of  $R(\zeta; q)$  leads to a better understanding of the mock theta functions.

Another universal mock theta function is  $R_2(\zeta; q)$ , the generating function of the  $M_2$ -rank of partitions without repeated odd parts, which appears among the tenth order mock theta functions. This also has an elegant combinatorial statement. The  $M_2$ -rank of a partition without repeated odd parts is given by taking the ceiling of the largest part divided by 2, and then subtracting the number of parts. This rank was introduced by Berkovich and Garvan in [3] and further studied by Lovejoy and Osburn in [16]. We let  $N_2(m, n)$  denote the number of partitions of  $n$  without repeated odd parts and  $M_2$ -rank  $m$ . We denote the generating function by

$$R_2(\zeta; q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) \zeta^m q^n.$$

In this article we improve the understanding of the  $M_2$ -rank as the holomorphic part of a harmonic Maass form. We determine a more precise formula for the transformation of the associated harmonic Maass form and do so on a larger group than previously known. To aid in determining dissection formulas  $R_2(e^{\frac{2\pi i}{\ell}}; q) = \sum_{r=0}^{\ell-1} q^r A_r(q^\ell)$ , we find additional harmonic Maass forms with holomorphic parts corresponding to the mock modular parts of the  $A_r(q^\ell)$ . Additionally we give lower bounds on the orders of the cusps for the holomorphic parts of the harmonic Maass forms.

Partition ranks have been studied in terms of harmonic Maass forms in a number of recent works, perhaps the most influential being [5] and other important works are [1, 6, 7, 17] of which the second and third deal with the  $M_2$ -rank. Recently Garvan [9] has made remarkable improvements to the transformations and levels from [5] for the harmonic Maass forms related to  $R(\zeta; q)$ . A key observation in Garvan's work is that on a rather large subgroup, the harmonic Maass form related to  $R(\zeta; q)$  has the same multiplier as  $\eta(\tau)^{-1}$ , which is the generating function for partitions. The same phenomenon will occur here. That is to say the harmonic Maass form related to  $R_2(\zeta; q)$  will transform like  $\frac{\eta(2\tau)}{\eta(4\tau)\eta(\tau)}$ , which is the generating function for partitions without repeated odd parts. This also occurs with the Dyson rank of overpartitions,  $\overline{R}(\zeta; q)$ , which the author studied in [13]. Although not stated explicitly in that article, one can easily verify that on a certain subgroup the transformation formula for the Dyson rank of overpartitions agrees with  $\frac{\eta(2\tau)}{\eta(\tau)^2}$ , the generating function for overpartitions. With an improved understanding of the  $M_2$ -rank, we give new identities for  $R_2(\zeta; q)$  when  $\zeta$  is set equal to certain roots of unity. In the next section we formally state our definitions and results, and then give an outline of the rest of the article.

For the reader that wishes to compare the various appearances of the the three rank functions mentioned here, we note the following. In the notation of Gordon and McIntosh [10], we have  $R(\zeta; q) = h_3(\zeta, q) = (1 - \zeta)(1 + \zeta g_3(\zeta, q))$ ,  $\overline{R}(\zeta; q) = (1 - \zeta)(1 - \zeta^{-1})g_2(\zeta, q)$ , and  $R_2(\zeta; q) = h_2(\zeta, -q)$ . In the notation of Hickerson and Mortenson [12], we have  $R(\zeta; q) = (1 - \zeta)(1 + \zeta g(\zeta, q))$ ,  $\overline{R}(\zeta; q) = (1 - \zeta)(1 - \zeta^{-1})h(\zeta, q)$ , and  $R_2(\zeta; q) = (1 - \zeta)\zeta^{\frac{1}{2}}k(\zeta^{\frac{1}{2}}, -q)$ .

## 2. STATEMENT OF RESULTS

As in the introduction we let  $N_2(m, n)$  denote the number of partitions of  $n$  without repeated odd parts with  $M_2$ -rank  $m$ . Furthermore we let  $N_2(k, c, n)$  denote the number of partitions of  $n$  without repeated odd parts with  $M_2$ -rank congruent to  $k$  modulo  $c$ . To state our definitions and results we use the standard product notation,

$$(\zeta; q)_n = \prod_{j=0}^{n-1} (1 - \zeta q^j), \quad (\zeta; q)_\infty = \prod_{j=0}^{\infty} (1 - \zeta q^j),$$

$$(\zeta_1, \dots, \zeta_k; q)_n = (\zeta_1; q)_n \dots (\zeta_k; q)_n, \quad (\zeta_1, \dots, \zeta_k; q)_\infty = (\zeta_1; q)_\infty \dots (\zeta_k; q)_\infty.$$

Also we let  $q = \exp(2\pi i\tau)$  for  $\tau \in \mathcal{H}$ , that is  $\text{Im}(\tau) > 0$ . For  $c$  a positive integer and  $a$  an integer, we let  $\zeta_c^a = \exp(2\pi i a/c)$ .

As noted in [16], one may use Theorem 1.2 of [15] to find that the generating function for  $N_2(m, n)$  is given by

$$\begin{aligned} R2(\zeta; q) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) \zeta^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(\zeta q^2, \zeta^{-1} q^2; q^2)_n} \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-\zeta)(1-\zeta^{-1})(-1)^n q^{2n^2+n}(1+q^{2n})}{(1-\zeta q^{2n})(1-\zeta^{-1} q^{2n})} \right). \end{aligned}$$

Next for  $a$  and  $c$  integers,  $c > 0$ , and  $c \nmid 2a$ , we define

$$\begin{aligned} \mathcal{R}2(a, c; \tau) &= q^{-\frac{1}{8}} R2(\zeta_c^a; e^{2\pi i\tau}), \\ \widetilde{\mathcal{R}2}(a, c; \tau) &= \mathcal{R}2(a, c; \tau) - i(1 - \zeta_c^a) \zeta_{2c}^{-a} \left( e^{\frac{\pi i}{4}} \int_{-\tau}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w) dw}{\sqrt{-i(w + \tau)}} + e^{-\frac{\pi i}{4}} \int_{-\tau}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w) dw}{\sqrt{-i(w + \tau)}} \right), \end{aligned}$$

where  $g_{a,b}(\tau)$  is a theta function of Zwegers [23], the definition of which we give in the next section. The factor of  $q^{-\frac{1}{8}}$  is necessary for certain transformations to work correctly. We notice that it is immediately clear that  $\mathcal{R}2(a + c, c; \tau) = \mathcal{R}2(a, c; \tau)$ ; it is also true that  $\widetilde{\mathcal{R}2}(a + c, c; \tau) = \widetilde{\mathcal{R}2}(a, c; \tau)$  and this follows immediately from properties of  $g_{a,b}(\tau)$ .

We will find  $R2$ ,  $\mathcal{R}2$ , and  $\widetilde{\mathcal{R}2}$  are related to the following functions. For  $k$  and  $c$  integers,  $0 \leq k < c$ , we define

$$\begin{aligned} S(k, c; \tau) &= \begin{cases} \frac{(-1)^{\frac{c+1}{2}} q^{-(c-k)(2c-2k-1)}}{(q^c, q^{4c^2-c}, q^{4c^2}; q^{4c^2})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2c^2 n(n+1)-cn}}{1 - q^{4c^2 n + (2k-c)2c}} & \text{if } c \text{ is odd,} \\ \frac{(-1)^{\frac{c}{2}} q^{-(c-k)(2c-2k-1)}}{(-1, -q^{4c^2}, q^{4c^2}; q^{4c^2})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2c^2 n(n+1)}}{1 - q^{4c^2 n + (1+4k-2c)c}} & \text{if } c \text{ is even,} \end{cases} \\ \mathcal{S}(k, c; \tau) &= i^{c^2} q^{-\frac{1}{8}} S(k, c; \tau) = \begin{cases} \frac{(-1)^{\frac{c+1}{2}} i q^{-\frac{(1+4k-4c)^2}{8}}}{(q^c, q^{4c^2-c}, q^{4c^2}; q^{4c^2})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2c^2 n(n+1)-cn}}{1 - q^{4c^2 n + (2k-c)2c}} & \text{if } c \text{ is odd,} \\ \frac{(-1)^{\frac{c}{2}} q^{-\frac{(1+4k-4c)^2}{8}}}{(-1, -q^{4c^2}, q^{4c^2}; q^{4c^2})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2c^2 n(n+1)}}{1 - q^{4c^2 n + (1+4k-2c)c}} & \text{if } c \text{ is even,} \end{cases} \\ \widetilde{\mathcal{S}}(k, c; \tau) &= \mathcal{S}(k, c; \tau) - (-1)^k i^{1+c} e^{-\frac{\pi i}{4}} c \int_{-\tau}^{i\infty} \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(4c^2 w) dw}{\sqrt{-i(w + \tau)}}. \end{aligned}$$

We will see in Section 3 that we can define the functions  $\mathcal{S}(k, c; \tau)$  and  $\widetilde{\mathcal{S}}(k, c; \tau)$  for all integer values of  $k$ , however some care must be taken in doing so as the above definitions would not be the proper choice and they do not satisfy the properties that  $\mathcal{S}(k + c, c; \tau) = \mathcal{S}(k, c; \tau)$  and  $\widetilde{\mathcal{S}}(k + c, c; \tau) = \widetilde{\mathcal{S}}(k, c; \tau)$ .

Our main theorem is the following. This theorem relates the  $M_2$ -rank generating function to harmonic Maass forms and modular forms.

**Theorem 2.1.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $t = \frac{c}{\gcd(4, c)}$ .*

- (1)  $\widetilde{\mathcal{R}2}(a, c; 8\tau)$  is a harmonic Maass form of weight  $1/2$  and  $\mathcal{R}2(a, c; 8\tau)$  is a mock modular of weight  $1/2$  on  $\Gamma_0(256t^2) \cap \Gamma_1(8t)$ .
- (2) The function

$$\widetilde{\mathcal{R}2}(a, c; \tau) - i^{-c} (1 - \zeta_c^a) \zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \widetilde{\mathcal{S}}(k, c; \tau)$$

is holomorphic in  $\tau$  and has at worst poles at the cusps. Furthermore,

$$\begin{aligned} \widetilde{\mathcal{R}2}(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \widetilde{\mathcal{S}}(k, c; \tau) \\ = \mathcal{R}2(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \mathcal{S}(k, c; \tau). \end{aligned}$$

(3) The function

$$\frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)} \left( \mathcal{R}2(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \mathcal{S}(k, c; \tau) \right)$$

is a weakly holomorphic modular form of weight 1 on  $\Gamma_0(4c^2 \cdot \gcd(c, 2)) \cap \Gamma_1(4c)$ .

(4) If  $c$  is odd then the function

$$\frac{\eta(4c^2\tau)\eta(c^2\tau)}{\eta(2c^2\tau)} \left( \mathcal{R}2(a, c; \tau) - i^{-c}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \mathcal{S}(k, c; \tau) \right)$$

is a weakly holomorphic modular form of weight 1 on  $\Gamma_0(4c^2) \cap \Gamma_1(4c)$ .

Part (1) of Theorem 2.1 is not entirely new. In [6] Bringmann, Ono, and Rhoades studied various general harmonic Maass forms, one of which has essentially  $R2(\zeta_c^a, 8t^2z)$  as the holomorphic part. However in this case the subgroup is  $\Gamma_1(2^{10}t^4)$ . Our construction of the harmonic Maass forms greatly differs from that of [6]. Additionally, Hickerson and Mortenson in [11] considered the function

$$D_2(a; M) = \sum_{n=0}^{\infty} \left( N_2(r, M, n) - \frac{p_o(n)}{t} \right) q^n,$$

where  $p_o(n)$  is the number of partitions without repeated odd parts. There they showed that  $D_2(a; M)$  can be expressed as the sum of two Appell-Lerch sums of a form similar to  $S(k, c; \tau)$  and a theta function.

In Section 5 we give various formulas so that we may determine the orders of these modular forms at cusps. With this we can then verify various identities for the  $M_2$ -rank function by passing to modular forms and using the valence formula. In particular one can use this to give an exact description of the  $c$ -dissection of  $R2(\zeta_c^a; q)$  in terms of generalized Lambert series and modular forms. We give these identities for  $c = 7$ . Similar formulas were determined by Lovejoy and Osburn for  $c = 3$  and  $c = 5$  in [16] and for  $c = 6$  and  $c = 10$  by Mao in [18]. Rather than using harmonic Maass forms, those formulas use the  $q$ -series techniques developed by Atkin and Swinnerton-Dyer [2] to determine rank difference formulas for the rank of partitions. Furthermore in [11] Hickerson and Mortenson demonstrated how their identities for  $D_2(a; M)$  can be used to prove the formulas for  $c = 3$  and  $c = 5$ . With this in mind, we see that there are at least three different proof techniques for determining the  $c$ -dissection of  $R2(\zeta_c^a; q)$ . One might ask for the pros and cons of the approach we use in this article. Our approach with harmonic Maass forms shares a difficulty with the other two methods, which is that we must correctly guess the identity before we can prove it. This approach with harmonic Maass forms helps us to guess the identity in that we know the weight and level of the modular forms.

**Theorem 2.2.** Let  $\zeta_7$  be a primitive seventh root of unity,  $J_a = (q^a, q^{28-a}; q^{28})_{\infty}$  for  $1 \leq a \leq 14$ ,  $J_0 = (q^{28}; q^{28})_{\infty}$ , and  $A(w, x, y) = w\zeta_7 + x\zeta_7^2 + y\zeta_7^3 + y\zeta_7^4 + x\zeta_7^5 + w\zeta_7^6$ . Then

$$R2(\zeta_7; q) = R2_0(q^7) + qR2_1(q^7) + q^2R2_2(q^7) + q^3R2_3(q^7) + q^4R2_4(q^7) + q^5R2_5(q^7) + q^6R2_6(q^7).$$

Here each  $R2_i(q)$  can be written as a sum of terms with  $S(k, 7; \frac{\tau}{7})$  and quotients of  $J_a$ . Due to the complexity of the expressions, we only state the definition of  $R2_0(q)$  here and give the definitions of the other  $R2_i(q)$  in Section 6. We have

$$\begin{aligned} R2_0(q) = & A(-3, -2, -2)S(0, 7; \tau/7) + A(-3, -2, -2)S(3, 7; \tau/7) + \frac{A(11, 15, 8)q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6J_9J_{11}J_{12}J_{13}J_{14}^2} \\ & + \frac{A(-11, -13, -9)q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}^2J_{12}J_{13}^2J_{14}} + \frac{A(1, 2, 0)q^{-3}J_0J_7^2J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}^2J_{13}^2J_{14}} + \frac{A(9, 11, 8)q^{-3}J_0J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2} + \frac{A(-14, -16, -10)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9^2J_{10}J_{11}^2J_{12}J_{13}J_{14}} \\ & + \frac{A(-1, 1, -3)q^{-3}J_0J_6J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}^2J_{14}^2} + \frac{A(-6, -4, -4)q^{-3}J_0J_6J_8^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}^2J_{11}^2J_{12}J_{13}^2J_{14}} + \frac{A(-12, -9, -6)q^{-3}J_0J_6J_9J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}J_{13}^2J_{14}^2} \end{aligned}$$

$$\begin{aligned}
& + \frac{A(-7,-9,-4)q^{-3}J_0J_6J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{12}J_{13}J_{14}} + \frac{A(-26,-19,-16)q^{-3}J_0J_6J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(8,6,5)q^{-3}J_0J_6J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}} + \frac{A(35,34,22)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-21,-19,-11)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-3,-3,-1)q^{-3}J_0J_6J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} + \frac{A(36,24,21)q^{-3}J_0J_6J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(16,17,10)q^{-3}J_0J_6J_7J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-5,-2,-3)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{13}J_{14}} + \frac{A(-3,-4,-3)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(-8,-6,-5)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{13}J_{14}} + \frac{A(27,24,19)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}} \\
& + \frac{A(-8,-4,-5)q^{-2}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(3,1,2)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{14}} + \frac{A(-12,-8,-6)q^{-3}J_0J_6J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{13}J_{14}} \\
& + \frac{A(-9,-6,-6)q^{-3}J_0J_6J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1,-3,-3)q^{-3}J_0J_6J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{13}J_{14}} + \frac{A(5,2,3)q^{-3}J_0J_6J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{13}J_{14}} + \frac{A(11,6,6)q^{-2}J_0J_6J_8^2J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-6,-4,-3)q^{-2}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(6,4,4)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{12}J_{13}J_{14}} + \frac{A(34,29,22)q^{-3}J_0J_6^2J_8J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-6,-4,-4)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-6,-4,-4)q^{-3}J_0J_6^2J_9J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_8J_{11}J_{13}J_{14}} + \frac{A(3,2,2)q^{-3}J_0J_6^2J_9J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_8J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-28,-25,-18)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}} + \frac{A(-27,-26,-15)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{12}J_{13}J_{14}} + \frac{A(5,5,1)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}} + \frac{A(1,-1,0)q^{-3}J_0J_6^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(2,4,1)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(6,4,4)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}} + \frac{A(3,2,2)q^{-1}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-8,-8,-5)q^{-3}J_0J_8^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_6J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(6,4,4)q^{-3}J_0J_8^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5J_7J_{11}J_{12}J_{13}J_{14}} + \frac{A(2,1,1)J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(3,1,2)qJ_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}}.
\end{aligned}$$

It is trivial to verify that the  $q^\alpha S(k, 7; \tau/7)$  appearing in the definitions of the  $R2_i(q)$  are all series with integral powers of  $q$ . One advantage to writing the identity in this form is that we can also read off formulas involving the rank differences,

$$R2_{r,s,c}(d; q) = \sum_{n=0}^{\infty} (N_2(r, c, cn + d) - N_2(s, c, cn + d)) q^n.$$

To explain this, we let  $\zeta_c = \zeta_c^a$  and use that  $1 + \zeta_c + \zeta_c^2 + \dots + \zeta_c^{c-1} = 0$  to find that

$$R2(\zeta_c; q) = \sum_{n=0}^{\infty} \sum_{r=0}^{c-1} N_2(r, c, n) \zeta_c^r q^n = \sum_{r=1}^{c-1} \zeta_c^r \sum_{d=0}^{c-1} q^d R2_{r,0,c}(d, q^c).$$

Since  $N_2(m, n) = N_2(-m, n)$ , we know  $R2_{r,0,c}(d, q^c) = R2_{c-r,0,c}(d, q^c)$ . When  $c$  is odd we have

$$R2(\zeta_c; q) = \sum_{r=1}^{\frac{c-1}{2}} (\zeta_c^r + \zeta_c^{c-r}) \sum_{d=0}^{c-1} q^d R2_{r,0,c}(d; q^c),$$

whereas for even  $c$  we have

$$R2(\zeta_c; q) = \zeta_c^{\frac{c}{2}} \sum_{d=0}^{c-1} q^d R2_{\frac{c}{2},0,c}(d; q^c) + \sum_{r=1}^{\frac{c-2}{2}} (\zeta_c^r + \zeta_c^{c-r}) \sum_{d=0}^{c-1} q^d R2_{r,0,c}(d; q^c).$$

In the case of  $c$  being an odd prime, we have that  $\zeta_c, \zeta_c^2, \dots, \zeta_c^{c-1}$  are linearly independent over  $\mathbb{Q}$ , so if

$$R2(\zeta_c; q) = \sum_{r=1}^{\frac{c-1}{2}} (\zeta_c^r + \zeta_c^{c-r}) \sum_{d=0}^{c-1} q^d S_r(d; q^c),$$

and each  $S_r(d; q)$  is a series in  $q$  with rational coefficients, then  $S_r(d; q) = R2_{r,0,c}(d; q)$ . As an example, from the formula for  $R2_0(q)$ , we have that

$$\begin{aligned}
R2_{1,0,7}(0; q) &= -3S(0, 7; \tau/7) - 3S(3, 7; \tau/7) + \frac{11q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} - \frac{11q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} \\
&+ \frac{q^{-3}J_0J_7^2J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{9q^{-3}J_0J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}} - \frac{14q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} \\
&+ \dots
\end{aligned}$$

We note it is in fact more convenient to verify the identities for the  $R2_{r,0,7}(d; q)$ , as then we are working with integer coefficients rather than coefficients from  $\mathbb{Z}[\zeta_7]$ . When  $c$  is composite, we can also deduce an identity based on the minimal polynomial for  $\zeta_c$ .

The rest of the article is organized as follows. In Section 3 we recall the basics of modular and harmonic Maass forms and introduce the functions studied by Zwegers in [23]; these functions allow us to relate our

functions to harmonic Maass forms. In Section 4 we work out the transformation formulas for our functions and prove Theorem 2.1. In Section 5 we give formulas for the orders at cusps. In Section 6 we use the results of Sections 4 and 5 to prove Theorem 2.2. We end by giving a few remarks in Section 7.

### 3. MODULAR FORMS, HARMONIC MAASS FORMS, AND ZWEGERS' FUNCTIONS

We first recall some basic terminology and results for modular and Maass forms. For further details, one may consult [5, 19, 20, 23]. We have that  $\mathrm{SL}_2(\mathbb{Z})$  is the multiplicative group of  $2 \times 2$  integer matrices with determinant 1. The principal congruence subgroup of level  $N$  is

$$\Gamma(N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \equiv \delta \equiv 1 \pmod{N}, \beta \equiv \gamma \equiv 0 \pmod{N} \right\}.$$

A subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$  is called a congruence subgroup if  $\Gamma \supseteq \Gamma(N)$  for some  $N$ . Three congruence subgroups we will use are

$$\begin{aligned} \Gamma_0(N) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv 0 \pmod{N} \right\}, \\ \Gamma_1(N) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \equiv \delta \equiv 1 \pmod{N}, \gamma \equiv 0 \pmod{N} \right\}, \\ \Gamma^0(N) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \beta \equiv 0 \pmod{N} \right\}. \end{aligned}$$

We recall  $\mathrm{SL}_2(\mathbb{Z})$  acts on  $\mathcal{H}$  via Mobius transformations, that is if  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  then  $A\tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$ . Additionally we let  $j(A, \tau) = \gamma\tau + \delta$ . One can verify that  $j(AB, \tau) = j(A, B\tau) \cdot j(B, \tau)$ .

We recall a weakly holomorphic modular form of integral weight  $k$  on a congruence subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$  is a holomorphic function  $f$  on  $\mathcal{H}$  such that

- (1) if  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ , then  $f(A\tau) = (\gamma\tau + \delta)^k f(\tau)$ ,
- (2) if  $B \in \mathrm{SL}_2(\mathbb{Z})$  then  $(B : \tau)^{-k} f(B\tau)$  has an expansion of the form  $\sum_{n=n_0}^{\infty} a_n \exp(2\pi i n z / N)$ , where  $n_0 \in \mathbb{Z}$  and  $N$  is a positive integer.

When  $k$  is a half integer, we require  $\Gamma \subset \Gamma_0(4)$  and replace (1) with  $f(A\tau) = \left(\frac{\gamma}{\delta}\right)^{2k} \epsilon(\delta)^{-2k} (\gamma\tau + \delta)^k f(\tau)$ . Here  $\left(\frac{m}{n}\right)$  is the Jacobi symbol extended to all integers  $n$  by

$$\begin{aligned} \left(\frac{0}{\pm 1}\right) &= 1, \\ \left(\frac{m}{n}\right) &= \begin{cases} \left(\frac{m}{|n|}\right) & \text{if } m > 0 \text{ or } n > 0, \\ -\left(\frac{m}{|n|}\right) & \text{if } m < 0 \text{ and } n < 0, \end{cases} \end{aligned}$$

and  $\epsilon(\delta)$  is 1 when  $\delta \equiv 1 \pmod{4}$  and is  $i$  otherwise.

A harmonic weak Maass form satisfies the transformation law in (1), but the condition of holomorphic is replaced with being smooth and annihilated by the weight  $k$  hyperbolic Laplacian operator,

$$\Delta_k = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

where  $\tau = x + iy$ , and condition (2) is replaced with  $(B : \tau)^{-k} f(B\tau)$  having at most linear exponential growth as  $\tau \rightarrow i\infty$ . We note that with  $\frac{\partial}{\partial \tau} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$  and  $\frac{\partial}{\partial \bar{\tau}} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$ , we have  $\Delta_k = -4y^{2-k} \frac{\partial}{\partial \tau} y^k \frac{\partial}{\partial \bar{\tau}}$ .

If  $f$  is a harmonic weak Maass form of weight  $2 - k$ , with  $k > 1$ , on  $\Gamma_1(N)$  and  $f$  satisfies additional growth conditions, then  $f$  can be written as  $f = f^+ + f^-$ , where  $f^+$  and  $f^-$  have expansions of the form

$$f^+(\tau) = \sum_{n=n_0}^{\infty} a(n) q^n, \quad f^-(\tau) = \sum_{n=1}^{\infty} b(n) \Gamma(k-1, 4\pi n y) q^{-n}.$$

Here  $\Gamma$  is the incomplete Gamma function given by  $\Gamma(y, x) = \int_x^{\infty} e^{-t} t^{y-1} dt$ . We call  $f^+$  the holomorphic part and  $f^-$  the non-holomorphic part. The non-holomorphic part is often written instead as an integral of

the form

$$f^-(\tau) = \int_{-\bar{\tau}}^{\infty} g(w)(-i(w+\tau))^{k-2} dw,$$

where  $g = \xi_{2-k}(f)$  and  $\xi_k = 2iy^k \frac{\partial}{\partial \bar{\tau}}$ . When  $f$  is a harmonic Maass form on some other congruence subgroup, similar expansions exist, but with  $q$  replaced by a fractional power of  $q$ .

We now recall the various functions needed for the Maass forms. For  $u, v, z \in \mathbb{C}$ ,  $\tau \in \mathcal{H}$ , and  $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$  we let

$$\begin{aligned} \vartheta(z; \tau) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} \exp\left(\pi i n^2 \tau + 2\pi i n \left(z + \frac{1}{2}\right)\right) = -iq^{\frac{1}{8}} e^{-\pi i z} (e^{2\pi i z}, e^{-2\pi i z} q, q; q)_{\infty}, \\ \mu(u, v; \tau) &= \frac{\exp(\pi i u)}{\vartheta(v; \tau)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \exp(\pi i n(n+1)\tau + 2\pi i n v)}{1 - \exp(2\pi i n \tau + 2\pi i u)}. \end{aligned}$$

Next for  $z \in \mathbb{C}$ ,  $y = \text{Im}(\tau)$ , and  $a = \text{Im}(z)/\text{Im}(\tau)$  we define

$$\begin{aligned} E(z) &= 2 \int_0^z \exp(-\pi w^2) dw, \\ R(z; \tau) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} \left( \text{sgn}(n) - E((n+a)\sqrt{2y}) \right) (-1)^{n-\frac{1}{2}} \exp(-\pi i n^2 \tau - 2\pi i n z). \end{aligned}$$

We remark that  $R(z; \tau)$  is not related to the rank generating function  $R(\zeta; q)$  discussed in the introduction. For  $a, b \in \mathbb{R}$  we set

$$g_{a,b}(\tau) = \sum_{n \in \mathbb{Z} + a} n \exp(\pi i n^2 \tau + 2\pi i n b).$$

Finally for  $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$  we set

$$\tilde{\mu}(u, v; \tau) = \mu(u, v; \tau) + \frac{i}{2} R(u-v; \tau).$$

In his revolutionary PhD thesis [23], Zwegers studied these functions and gave their transformation formulas which are essential to our results.

**Proposition 3.1.** *Let  $\zeta = \exp(\pi i u)$ , then*

$$R2(\zeta; \tau) = i(1 - \zeta) \left( \zeta^{-1} \mu(u, -\tau; 4\tau) - \mu(u, \tau; 4\tau) \right).$$

*Proof.* We have

$$\begin{aligned} R2(\zeta; \tau) &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-\zeta)(1-\zeta^{-1})(-1)^n q^{2n^2+n}(1+q^{2n})}{(1-\zeta q^{2n})(1-\zeta^{-1} q^{2n})} \right) \\ &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} (-1)^n q^{2n^2+n} \left( \frac{(1-\zeta)}{(1-\zeta q^{2n})} + \frac{(1-\zeta^{-1})}{(1-\zeta^{-1} q^{2n})} \right) \right) \\ &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}(1-\zeta)(1+\zeta q^{2n})}{(1-\zeta^2 q^{4n})} \\ &= (1-\zeta) \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}}{(1-\zeta^2 q^{4n})} + (1-\zeta)\zeta \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+3n}}{(1-\zeta^2 q^{4n})}. \end{aligned}$$

We note that

$$\frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} = \frac{(q^2; q^4)_{\infty}}{(q, q^2; q^2)_{\infty}} = \frac{1}{(q, q^3, q^4; q^4)_{\infty}},$$

and

$$\vartheta(\tau; 4\tau) = -i (q, q^3, q^4; q^4)_{\infty}.$$

Thus

$$\mu(u, \tau; 4\tau) = i\zeta \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+3n}}{1 - \zeta^2 q^{4n}},$$

and

$$\mu(u, -\tau; 4\tau) = -i\zeta \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1 - \zeta^2 q^{4n}}.$$

With this we see

$$R2(\zeta; \tau) = i\zeta^{-1}(1 - \zeta)\mu(u, -\tau; 4\tau) - i(1 - \zeta)\mu(u, \tau; 4\tau).$$

□

We note that we could also use functions from [22], rather than  $\mu(u, v; \tau)$ . In particular, the modular transformation and elliptic properties of functions of the form

$$e^{\pi i u} \sum_{n=-\infty}^{\infty} \frac{(-1)^{mn} q^{\frac{m(n+1)}{2}} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n},$$

for  $m \in \mathbb{Z}$  are well understood.

**Proposition 3.2.** *For  $a$  and  $c$  integers,  $c > 0$ , and  $c \nmid 2a$ , we have that*

$$\begin{aligned} \mathcal{R}2(a, c; \tau) &= i(1 - \zeta_c^a) \left( \zeta_c^{-a} q^{-\frac{1}{8}} \mu\left(\frac{2a}{c}, -\tau; 4\tau\right) - q^{-\frac{1}{8}} \mu\left(\frac{2a}{c}, \tau; 4\tau\right) \right), \\ \widetilde{\mathcal{R}}2(a, c; \tau) &= i(1 - \zeta_c^a) \left( \zeta_c^{-a} q^{-\frac{1}{8}} \tilde{\mu}\left(\frac{2a}{c}, -\tau; 4\tau\right) - q^{-\frac{1}{8}} \tilde{\mu}\left(\frac{2a}{c}, \tau; 4\tau\right) \right). \end{aligned}$$

*Proof.* We see that the first identity is a trivial corollary of the previous proposition. For the second identity, given the definitions of  $\widetilde{\mathcal{R}}2$  and  $\tilde{\mu}$ , we see we must show that

$$\begin{aligned} & i(1 - \zeta_c^a) \left( \zeta_c^{-a} q^{-\frac{1}{8}} \frac{i}{2} R\left(\frac{2a}{c} + \tau; 4\tau\right) - q^{-\frac{1}{8}} \frac{i}{2} R\left(\frac{2a}{c} - \tau; 4\tau\right) \right) \\ &= -i(1 - \zeta_c^a) \zeta_{2c}^{-a} \left( e^{\frac{\pi i}{4}} \int_{-\tau}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w) dw}{\sqrt{-i(w + \tau)}} + e^{-\frac{\pi i}{4}} \int_{-\tau}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w) dw}{\sqrt{-i(w + \tau)}} \right). \end{aligned}$$

However, by Theorem 1.16 of [23] we have that

$$R\left(\frac{2a}{c} - \tau; 4\tau\right) = -\exp\left(\frac{4\pi i \tau}{16} + \frac{2\pi i}{4}\left(\frac{-2a}{c} + \frac{1}{2}\right)\right) \int_{-4\tau}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(z) dz}{\sqrt{-i(z + 4\tau)}} = -2q^{\frac{1}{8}} e^{\frac{\pi i}{4}} \zeta_{2c}^{-a} \int_{-\tau}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w) dw}{\sqrt{-i(w + \tau)}}.$$

Similarly we have

$$R\left(\frac{2a}{c} + \tau; 4\tau\right) = -\exp\left(\frac{4\pi i \tau}{16} - \frac{2\pi i}{4}\left(\frac{-2a}{c} + \frac{1}{2}\right)\right) \int_{-4\tau}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(z) dz}{\sqrt{-i(z + 4\tau)}} = -2q^{\frac{1}{8}} e^{-\frac{\pi i}{4}} \zeta_{2c}^a \int_{-\tau}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w) dw}{\sqrt{-i(w + \tau)}}.$$

Thus the proposition follows. □

**Proposition 3.3.** *Suppose  $k$  and  $c$  are integers and  $0 \leq k < c$ . Then*

$$\begin{aligned} \mathcal{S}(k; c; \tau) &= \begin{cases} q^{-\frac{(1+4k-2c)^2}{8}} \mu((2k-c)2c\tau, \frac{c-1}{2} - c\tau; 4c^2\tau) & \text{if } c \text{ is odd,} \\ q^{-\frac{(1+4k-2c)^2}{8}} \mu((1+4k-2c)c\tau, \frac{c-1}{2}; 4c^2\tau) & \text{if } c \text{ is even,} \end{cases} \\ \widetilde{\mathcal{S}}(k; c; \tau) &= \begin{cases} q^{-\frac{(1+4k-2c)^2}{8}} \tilde{\mu}((2k-c)2c\tau, \frac{c-1}{2} - c\tau; 4c^2\tau) & \text{if } c \text{ is odd,} \\ q^{-\frac{(1+4k-2c)^2}{8}} \tilde{\mu}((1+4k-2c)c\tau, \frac{c-1}{2}; 4c^2\tau) & \text{if } c \text{ is even.} \end{cases} \end{aligned}$$

*Proof.* When  $c$  is odd, we have that

$$q^{-\frac{(1+4k-2c)^2}{8}} \mu\left((2k-c)2c\tau, \frac{c-1}{2} - c\tau; 4c^2\tau\right) = \frac{q^{-\frac{(1+4k-2c)^2}{8} + (2k-c)c}}{\vartheta\left(\frac{c-1}{2} - c\tau; 4c^2\tau\right)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2c^2n(n+1)-cn}}{1 - q^{4c^2n+(2k-c)2c}}$$



$$\begin{aligned}
&= \frac{(-1)^{\frac{c+1}{2}} i q^{-\frac{(1+4k-4c)^2}{8}}}{(q^c, q^{4c^2-c}, q^{4c^2}, q^{4c^2})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2c^2 n(n+1)-cn}}{1 - q^{4c^2 n + (2k-2c)2c}} \\
&= \mathcal{S}(k, c; \tau).
\end{aligned}$$

When  $c$  is even, we instead have that

$$\begin{aligned}
q^{-\frac{(1+4k-2c)^2}{8}} \mu((1+4k-2c)c\tau, \frac{c-1}{2}, 4c^2\tau) &= \frac{q^{-\frac{(1+4k-2c)^2}{8} + \frac{(1+4k-2c)c}{2}}}{\vartheta(\frac{c-1}{2}, 4c^2\tau)} \sum_{n=-\infty}^{\infty} \frac{q^{2c^2 n(n+1)}}{1 - q^{4c^2 n + (1+4k-2c)c}} \\
&= \frac{(-1)^{\frac{c}{2}} q^{-\frac{(1+4k-4c)^2}{8}}}{(-1, -q^{4c^2}, q^{4c^2}, q^{4c^2})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{2c^2 n(n+1)}}{1 - q^{4c^2 n + (1+4k-2c)c}} \\
&= \mathcal{S}(k, c; \tau).
\end{aligned}$$

This establishes the identities for  $\mathcal{S}(k, c; \tau)$ . To verify the identities for  $\tilde{\mathcal{S}}(k, c; \tau)$ , we need only verify that for  $0 \leq k < c$  we have

$$\frac{i}{2} q^{-\frac{(1+4k-2c)^2}{8}} R((1+4k-2c)c\tau - \frac{c-1}{2}, 4c^2\tau) = (-1)^{k+1} i^{1+c} e^{-\frac{\pi i}{4} c} \int_{-\bar{\tau}}^{\infty} \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(4c^2 w)}{\sqrt{-i(w+\tau)}} dw.$$

For this we use Theorem 1.6 of [23] to determine that

$$\begin{aligned}
&\frac{i}{2} q^{-\frac{(1+4k-2c)^2}{8}} R((1+4k-2c)c\tau - \frac{c-1}{2}, 4c^2\tau) \\
&= -\frac{i}{2} q^{-\frac{(1+4k-2c)^2}{8}} \exp\left(\pi i \tau \frac{(1+4k-2c)^2}{4} - \pi i \frac{(1+4k-2c)}{4}\right) \int_{-4c^2\tau}^{i\infty} \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(z)}{\sqrt{-i(z+4c^2\tau)}} dz \\
&= (-1)^{k+1} i^{1+c} e^{-\frac{\pi i}{4} c} \int_{-\bar{\tau}}^{i\infty} \frac{g_{\frac{1+4k}{4c}, \frac{c}{2}}(4c^2 w)}{\sqrt{-i(4c^2 w + 4c^2\tau)}} dw.
\end{aligned}$$

□

It is through these identities that we take the definitions of  $\mathcal{S}(k, c; \tau)$  and  $\tilde{\mathcal{S}}(k, c; \tau)$  when  $k$  is an arbitrary integer; that is to say they are defined in terms of  $\mu$  and  $\tilde{\mu}$  rather than series and integrals. Additionally we see the definitions depends on the parity of  $c$  because  $\mu(u, v; \tau)$  and  $\tilde{\mu}(u, v; \tau)$  require  $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$ . One can easily check that  $\tilde{\mathcal{S}}(k+c, c; \tau) = (-1)^c \tilde{\mathcal{S}}(k, c; \tau)$ , however the relation for  $\mathcal{S}(k+c, c; \tau)$  is not as elegant. In particular one can verify that  $\mathcal{S}(k+c, c; \tau) = i^{-c} q^{\frac{(1+4k+c)c}{2}} + (-1)^c \mathcal{S}(k, c; \tau)$ . Lastly we note when  $c$  is odd we can rewrite  $\tilde{\mathcal{S}}(k, c; \tau)$  as

$$\tilde{\mathcal{S}}(k, c; \tau) = (-1)^{\frac{c-1}{2}} q^{-\frac{(1+4k-2c)^2}{8}} \tilde{\mu}((2k-c)2c\tau, -c\tau; 4c^2\tau).$$

Working with  $\tilde{\mu}(u, v; \tau)$  is advantageous in that the transformation under the action of the modular group  $\text{SL}_2(\mathbb{Z})$  is known and quite elegant. With this we are able to determine a simple multiplier for  $\tilde{\mathcal{R}2}(a, c; \tau)$  and  $\tilde{\mathcal{S}}(k, c; \tau)$ . For this reason we do not need to replace  $\tau$  by  $8\tau$  to have a strong understanding of these functions. Additionally the transformations of  $\tilde{\mu}$  allow us to determine the behavior of  $\tilde{\mathcal{R}2}(a, c; \tau)$  and  $\tilde{\mathcal{S}}(k, c; \tau)$  at the cusps.

#### 4. TRANSFORMATIONS FORMULAS

For a matrix  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , we have,  $\nu(A)$ , the  $\eta$ -multiplier defined by

$$\eta(A\tau) = \nu(A) \sqrt{\gamma\tau + \delta} \eta(\tau),$$

where  $\eta(\tau)$  is Dedekind's eta-function,

$$\eta(\tau) = q^{\frac{1}{24}} (q; q)_\infty.$$

A convenient form for the  $\eta$ -multiplier when  $\gamma \neq 0$ , which can be found as Theorem 2 in Chapter 4 of [14], is

$$\nu(A) = \begin{cases} \left(\frac{\delta}{|\gamma|}\right) \exp\left(\frac{\pi i}{12} ((\alpha + \delta)\gamma - \beta\delta(\gamma^2 - 1) - 3\gamma)\right) & \text{if } \gamma \equiv 1 \pmod{2}, \\ \left(\frac{\delta}{2}\right) \exp\left(\frac{\pi i}{12} ((\alpha + \delta)\gamma - \beta\delta(\gamma^2 - 1) + 3\delta - 3 - 3\gamma\delta)\right) & \text{if } \delta \equiv 1 \pmod{2}. \end{cases} \quad (4.1)$$

For an integer  $m$  we let

$$A_m = \begin{pmatrix} \alpha & m\beta \\ \gamma/m & \delta \end{pmatrix}.$$

The utility of  $A_m$  is in the fact that  $mA\tau = A_m(m\tau)$ .

Our transformation formulas for  $\widetilde{\mathcal{R}2}(a, c; \tau)$  and  $\widetilde{\mathcal{S}}(k, c; \tau)$  are easily deduced by the transformations of  $\tilde{\mu}(u, v; \tau)$ . The following essential properties are from Theorem 1.11 of [23]. If  $k, l, m, n$  are integers then

$$\tilde{\mu}(u + k\tau + l, v + m\tau + n; \tau) = (-1)^{k+l+m+n} \exp(\pi i \tau (k-m)^2 + 2\pi i (k-m)(u-v)) \tilde{\mu}(u, v; \tau). \quad (4.2)$$

If  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  then

$$\tilde{\mu}\left(\frac{u}{\gamma\tau + \delta}, \frac{v}{\gamma\tau + \delta}; A\tau\right) = \nu(A)^{-3} \exp\left(-\frac{\pi i \gamma (u-v)^2}{\gamma\tau + \delta}\right) \sqrt{\gamma\tau + \delta} \tilde{\mu}(u, v; \tau). \quad (4.3)$$

The propositions and formulas of this section are organized as follows. We begin with  $\widetilde{\mathcal{R}2}(a, c; \tau)$  by first investigating the most general transformation that follows from (4.3), we then refine this result by restricting to smaller congruence subgroups of  $\text{SL}_2(\mathbb{Z})$  until we have a subgroup that yields a simple multiplier. We additionally need a formula for the transformation of  $\widetilde{\mathcal{R}2}(a, c; \tau)$  under the full modular group to deduce that  $\widetilde{\mathcal{R}2}(a, c; \tau)$  behaviors correctly at the cusps and to determine the orders at the cusps. We then verify  $\widetilde{\mathcal{R}2}(a, c; \tau)$  is annihilated by  $\Delta_{\frac{1}{2}}$ . With this we can deduce that replacing  $\tau$  by  $8\tau$  yields a harmonic Maass form of weight  $\frac{1}{2}$  on a certain congruence subgroup, however we find it beneficial to work with the original  $\widetilde{\mathcal{R}2}(a, c; \tau)$  and the subgroup on which it has a simple multiplier. In particular we find this multiplier agrees with that of a certain eta-quotient, which allows us to obtain functions that transform like weight 1 forms but that are not necessarily annihilated by  $\Delta_{\frac{1}{2}}$  or  $\Delta_1$ . We then determine a formula for the non-holomorphic part of  $\widetilde{\mathcal{R}2}(a, c; \tau)$  that allows us to determine the non-holomorphic parts of the  $c$ -dissections of  $\widetilde{\mathcal{R}2}(a, c; \tau)$ . We then introduce  $\widetilde{\mathcal{S}}(k, c; \tau)$  as the functions whose non-holomorphic parts agree with those of the  $c$ -dissections of  $\widetilde{\mathcal{R}2}(a, c; \tau)$ . After this we give  $\widetilde{\mathcal{S}}(k, c; \tau)$  the same treatment as  $\widetilde{\mathcal{R}2}(a, c; \tau)$ . Lastly we quickly deduce the necessary transformation formulas for various generalized eta quotients using the work of Biagioli in [4]. To begin we will find it convenient to define the functions

$$\widetilde{M}_{\pm}(a, c; \tau) = q^{-\frac{1}{32}} \tilde{\mu}\left(\frac{2a}{c}, \pm \frac{\tau}{4}; \tau\right).$$

**Proposition 4.1.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4)$ , then*

$$\begin{aligned} & \widetilde{M}_{\pm}(a, c; 4A\tau) \\ &= (-1)^{\beta} \zeta_{2c^2}^{-a^2\gamma\delta} \zeta_{2c}^{\pm a\gamma\beta} \exp\left(\frac{-\pi i \alpha \beta}{4}\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} q^{-\frac{1}{8}(\alpha \mp \frac{2a\gamma}{c})^2} \tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm \alpha\tau; 4\tau\right). \end{aligned}$$

*Proof.* We note that  $A_4 \in \text{SL}_2(\mathbb{Z})$  and so by (4.3) we have that

$$\begin{aligned} & \widetilde{M}_{\pm}(a, c; 4A\tau) \\ &= \exp\left(\frac{-\pi i}{4} A\tau\right) \tilde{\mu}\left(\frac{2a}{c}, \pm A\tau; A_4(4\tau)\right) \\ &= \exp\left(\frac{-\pi i}{4} A\tau\right) \tilde{\mu}\left(\frac{2a}{c} \frac{\gamma\tau + \delta}{\gamma\tau + \delta}, \frac{\pm \alpha\tau \pm \beta}{\gamma\tau + \delta}; \frac{\alpha(4\tau) + 4\beta}{4(\gamma\tau + \delta)}\right) \\ &= \exp\left(\frac{-\pi i}{4} A\tau\right) \exp\left(\frac{-\pi i \gamma}{4(\gamma\tau + \delta)} \left(\frac{2a}{c}(\gamma\tau + \delta) - (\pm \alpha\tau \pm \beta)\right)^2\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm \alpha\tau \pm \beta; 4\tau\right) \\ &= \exp\left(\frac{-\pi i}{4} A\tau\right) \exp\left(\frac{-\pi i \gamma (\alpha\tau + \beta)^2}{4(\gamma\tau + \delta)}\right) \exp\left(\frac{-\pi i \gamma}{4} \left(\frac{4a^2}{c^2}(\gamma\tau + \delta) \mp \frac{4a}{c}(\alpha\tau + \beta)\right)\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \\ & \quad \cdot \tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm \alpha\tau \pm \beta; 4\tau\right). \end{aligned}$$

However, since  $\alpha\delta - \beta\gamma = 1$ , we see that

$$\begin{aligned} \exp\left(\frac{-\pi i}{4} A\tau\right) \exp\left(\frac{-\pi i \gamma (\alpha\tau + \beta)^2}{4(\gamma\tau + \delta)}\right) &= \exp\left(\frac{-\pi i (\alpha\tau + \beta)}{4(\gamma\tau + \delta)} (1 + \alpha\gamma\tau + \beta\gamma)\right) \\ &= \exp\left(\frac{-\pi i (\alpha\tau + \beta)}{4(\gamma\tau + \delta)} (\alpha\gamma\tau + \alpha\delta)\right) \end{aligned}$$

$$= \exp\left(\frac{-\pi i(\alpha^2\tau + \alpha\beta)}{4}\right).$$

Thus

$$\begin{aligned} & \exp\left(\frac{-\pi i}{4}A\tau\right) \tilde{\mu}\left(\frac{2a}{c}, \pm A\tau; 4A\tau\right) \\ &= \exp\left(\frac{-\pi i\tau}{4}\left(\alpha^2 + \frac{4a^2\gamma^2}{c^2} \mp \frac{4a\alpha\gamma}{c}\right)\right) \exp\left(\frac{-\pi i}{4}\left(\alpha\beta + \frac{4a^2\gamma\delta}{c^2} \mp \frac{4a\beta\gamma}{c}\right)\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \\ & \quad \cdot \tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm\alpha\tau \pm \beta; 4\tau\right) \\ &= q^{-\frac{1}{8}\left(\alpha \mp \frac{2a\gamma}{c}\right)^2} \exp\left(\frac{-\pi i\alpha\beta}{4}\right) \zeta_{2c^2}^{-a^2\gamma\delta} \zeta_{2c}^{\pm a\beta\gamma} \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm\alpha\tau \pm \beta; 4\tau\right) \\ &= (-1)^\beta q^{-\frac{1}{8}\left(\alpha \mp \frac{2a\gamma}{c}\right)^2} \exp\left(\frac{-\pi i\alpha\beta}{4}\right) \zeta_{2c^2}^{-a^2\gamma\delta} \zeta_{2c}^{\pm a\beta\gamma} \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm\alpha\tau; 4\tau\right), \end{aligned}$$

where the last equality follows by (4.2).  $\square$

The rest of the transformation formulas have similar proofs. We apply (4.2) and (4.3) and reduce the various powers of  $q$  and roots of unity. For this reason we will omit the majority of these calculations as they are routine and straightforward, but rather long.

**Proposition 4.2.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2c) \cap \Gamma_1(4) \cap \Gamma_1(c)$ , then*

$$\widetilde{M}_\pm(a, c; 4A\tau) = (-1)^{\beta + \frac{a\gamma}{2c} + \frac{(\alpha-1)}{4} + \frac{a(\alpha-1)}{c} + \frac{\beta(\alpha-1)}{4}} \zeta_{2c^2}^{a^2\gamma} \exp\left(\frac{-\pi i\beta}{4}\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \widetilde{M}_\pm(a, c; 4\tau).$$

*Proof.* Using (4.2) we find that

$$\tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm\alpha\tau; 4\tau\right) = (-1)^{\frac{a\gamma}{2c} + \frac{(\alpha-1)}{4} + \frac{a(\alpha-1)}{c}} \zeta_{c^2}^{a^2\gamma} q^{\frac{1}{2}\left(\frac{a^2\gamma^2}{c^2} \mp \frac{a\gamma(\alpha-1)}{c} + \frac{(\alpha-1)^2}{4} \mp \frac{a\gamma}{c} + \frac{\alpha-1}{2}\right)} \tilde{\mu}\left(\frac{2a}{c}, \pm\tau; 4\tau\right).$$

With Proposition 4.1 we then have that

$$\widetilde{M}_\pm(a, c; 4A\tau) = (-1)^{\beta + \frac{a\gamma}{2c} + \frac{(\alpha-1)}{4} + \frac{a(\alpha-1)}{c}} \zeta_{c^2}^{a^2\gamma} \zeta_{2c^2}^{-a^2\gamma\delta} \zeta_{2c}^{\pm a\gamma\beta} \exp\left(\frac{-\pi i\alpha\beta}{4}\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \widetilde{M}_\pm(a, c; 4\tau).$$

Since  $A \in \Gamma_0(2c) \cap \Gamma_1(4) \cap \Gamma_1(c)$ , we deduce that

$$\zeta_{c^2}^{a^2\gamma} \zeta_{2c^2}^{-a^2\gamma\delta} = \zeta_{2c^2}^{a^2\gamma}, \quad \zeta_{2c}^{\pm a\gamma\beta} = 1, \quad \exp\left(\frac{-\pi i\alpha\beta}{4}\right) = (-1)^{\frac{\beta(\alpha-1)}{4}} \exp\left(\frac{-\pi i\beta}{4}\right),$$

which completes the proof of the proposition.  $\square$

**Corollary 4.3.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2c) \cap \Gamma_1(4) \cap \Gamma_1(c)$ , then*

$$\widetilde{\mathcal{R}2}(a, c; A\tau) = (-1)^{\beta + \frac{a\gamma}{2c} + \frac{(\alpha-1)}{4} + \frac{a(\alpha-1)}{c} + \frac{\beta(\alpha-1)}{4}} \zeta_{2c^2}^{a^2\gamma} \exp\left(\frac{-\pi i\beta}{4}\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \widetilde{\mathcal{R}2}(a, c; \tau).$$

*Proof.* This corollary follows immediately from Proposition 4.2 upon recalling that

$$\widetilde{\mathcal{R}2}(a, c; \tau) = i(1 - \zeta_c^a) \left( \zeta_c^{-a} q^{-\frac{1}{8}} \tilde{\mu}\left(\frac{2a}{c}, -\tau; 4\tau\right) - q^{-\frac{1}{8}} \tilde{\mu}\left(\frac{2a}{c}, \tau; 4\tau\right) \right).$$

$\square$

**Corollary 4.4.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2c^2) \cap \Gamma_0(4c) \cap \Gamma_1(4) \cap \Gamma_1(2c)$ , then*

$$\widetilde{\mathcal{R}2}(a, c; A\tau) = (-1)^{\beta + \frac{(\alpha-1)}{4} + \frac{\beta(\alpha-1)}{4}} \exp\left(\frac{-\pi i\beta}{4}\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \widetilde{\mathcal{R}2}(a, c; \tau).$$

To understand the behavior of  $\widetilde{\mathcal{R}2}(a, c; \tau)$  at the cusps of  $\text{SL}_2(\mathbb{Z})$ , we need the following transformation formula.

**Proposition 4.5.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , then*

$$\widetilde{M}_\pm(a, c; A\tau) = \zeta_{c^2}^{-2a^2\gamma\delta} \zeta_{2c}^{\pm a\beta\gamma} \exp\left(\frac{-\pi i\alpha\beta}{16}\right) \nu(A)^{-3} \sqrt{\gamma\tau + \delta} q^{-\frac{\alpha^2}{32} - \frac{2a^2\gamma^2}{c^2} \pm \frac{a\alpha\gamma}{2c}} \tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm\frac{\alpha}{4}\tau \pm \frac{\beta}{4}; \tau\right).$$

*Proof.* By (4.3) we have that

$$\begin{aligned}
& \widetilde{M}_{\pm}(a, c; A\tau) \\
&= \exp\left(-\frac{\pi i A\tau}{16}\right) \tilde{\mu}\left(\frac{2a}{c}, \pm \frac{A\tau}{4}; A\tau\right) \\
&= \exp\left(-\frac{\pi i A\tau}{16}\right) \exp\left(-\frac{\pi i \gamma}{\gamma\tau + \delta} \left(\frac{2a(\gamma\tau + \delta)}{c} \mp \frac{(\alpha\tau + \beta)}{4}\right)^2\right) \nu(A)^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{2a(\gamma\tau + \delta)}{c}, \pm \frac{(\alpha\tau + \beta)}{4}; \tau\right) \\
&= q^{-\frac{\alpha^2}{32} - \frac{2a^2\gamma^2}{c^2} \pm \frac{a\alpha\gamma}{2c}} \zeta_{c^2}^{-2a^2\gamma\delta} \zeta_{2c}^{\pm a\beta\gamma} \exp\left(-\frac{\pi i \alpha\beta}{16}\right) \nu(A)^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{2a\gamma}{c}\tau + \frac{2a\delta}{c}, \pm \frac{\alpha}{4}\tau \pm \frac{\beta}{4}; \tau\right).
\end{aligned}$$

□

**Proposition 4.6.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Then  $\widetilde{\mathcal{R}2}(a, c; \tau)$  has at worst linear exponential growth at the cusps and is annihilated by  $\Delta_{\frac{1}{2}}$ .*

*Proof.* To see that  $\widetilde{\mathcal{R}2}(a, c; \tau)$  has at worst linear exponential growth at the cusps, we just need to check that  $\widetilde{M}_{\pm}(a, c; 4\tau)$  has at worst linear exponential growth at the cusps. For this we use Proposition 4.5.

Suppose  $\alpha/\gamma$  is in reduced form, so we can then take  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ . We set  $g = \gcd(4, \gamma)$ ,  $x = 4\alpha/g$ ,  $u = \gamma/g$ , so that  $x$  and  $u$  are relatively prime. We then take  $L = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ . Next we set

$$B = L^{-1} \begin{pmatrix} 4\alpha & 4\beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} g & 4\beta v - \delta y \\ 0 & 4/g \end{pmatrix},$$

so that  $4A\tau = LB\tau$  and  $(\gamma\tau + \delta) = j(LB, \tau) = j(L, B\tau) \cdot j(B, \tau) = j(L, B\tau) \cdot \frac{4}{g}$ . With this we have

$$\begin{aligned}
& (\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{M}_{\pm}(a, c; 4A\tau) \\
&= (\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{M}_{\pm}(a, c; LB\tau) \\
&= \frac{\sqrt{g}}{2} \zeta_{c^2}^{-2a^2uv} \zeta_{2c}^{\pm a\gamma u} \exp\left(-\frac{\pi i xy}{16}\right) \nu(L)^{-3} \exp\left(-\pi i B\tau \left(\frac{x^2}{16} + \frac{4a^2u}{c^2} \mp \frac{axy}{c}\right)\right) \tilde{\mu}\left(\frac{2au}{c}B\tau + \frac{2av}{c}, \pm \frac{x}{4}B\tau \pm \frac{y}{4}; B\tau\right).
\end{aligned}$$

Noting

$$B\tau = \frac{g\tau + (4\beta v - \delta y)}{4/g} = \frac{g^2}{4}\tau + \frac{g(4\beta v - \delta y)}{4},$$

we have  $e^{\pi i B\tau} = \epsilon q^{\frac{g^2}{8}}$ , where  $|\epsilon| = 1$ , from which we see  $(\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{M}_{\pm}(a, c; 4A\tau)$  has at worst linear exponential growth as  $\tau \rightarrow i\infty$  by the definition of  $\tilde{\mu}$ .

Since  $\mathcal{R}2(a, c; \tau)$  is holomorphic in  $\tau$ , we have  $\Delta_{\frac{1}{2}}\mathcal{R}2(a, c; \tau) = 0$  and so we need only verify that  $\Delta_{\frac{1}{2}}q^{-\frac{1}{8}}R(\frac{2a}{c} \pm \tau; 4\tau) = 0$ . This follows quickly from Lemma 1.8 of [23] which states that

$$\frac{\partial}{\partial \bar{\tau}} R(a\tau - b; \tau) = -\frac{ie^{-2\pi a^2 y}}{\sqrt{2y}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} (-1)^{n-\frac{1}{2}} (n+a) e^{-\pi i n^2 \bar{\tau} - 2\pi i n(a\bar{\tau} + b)},$$

where  $y = \text{Im}(\tau)$ . By this Lemma we have that

$$\sqrt{y} \frac{\partial}{\partial \bar{\tau}} q^{-\frac{1}{8}} R(\pm\tau + \frac{2a}{c}; 4\tau) = -\sqrt{2}ie^{-\frac{\pi i \tau}{4} - \frac{2\pi 4y}{16}} A_{\pm}(\bar{\tau}) = -\sqrt{2}ie^{-\frac{\pi i \bar{\tau}}{4}} A_{\pm}(\bar{\tau}),$$

where  $A_{\pm}(\bar{\tau})$  is holomorphic in  $\bar{\tau}$ . Thus  $\sqrt{y} \frac{\partial}{\partial \bar{\tau}} q^{-\frac{1}{8}} R(\pm\tau + \frac{2a}{c}; 4\tau)$  is holomorphic in  $\bar{\tau}$  and so applying  $\frac{\partial}{\partial \tau}$  yields zero. That is to say,  $\Delta_{\frac{1}{2}}q^{-\frac{1}{8}}R(\frac{2a}{c} \pm \tau; 4\tau) = 0$ . □

**Corollary 4.7.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Then  $\widetilde{\mathcal{R}2}(a, c; 8\tau)$  is a weight  $\frac{1}{2}$  harmonic weak Maass form on  $\Gamma_0(256) \cap \Gamma_0(16c^2) \cap \Gamma_0(32c) \cap \Gamma_1(8) \cap \Gamma_1(2c)$ .*

*Proof.* We need only verify that for  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(256) \cap \Gamma_0(16c^2) \cap \Gamma_0(32c) \cap \Gamma_1(8) \cap \Gamma_1(2c)$  we have

$$\widetilde{\mathcal{R}2}(a, c; 8A\tau) = \left(\frac{\gamma}{\delta}\right) \sqrt{\gamma\tau + \delta} \widetilde{\mathcal{R}2}(a, c; 8\tau).$$

Since  $A_8 \in \Gamma_0(32) \cap \Gamma_0(2c^2) \cap \Gamma_0(4c) \cap \Gamma_1(8) \cap \Gamma_1(2c) \cap \Gamma^0(8)$ , Corollary 4.4 yields that

$$\widetilde{\mathcal{R}2}(a, c; 8A\tau) = \widetilde{\mathcal{R}2}(a, c; A_8(8\tau)) = (-1)^{\frac{\alpha-1}{4}} \nu(A_{32})^{-3} \sqrt{\gamma\tau + \delta} \widetilde{\mathcal{R}2}(a, c; 8\tau).$$

We note  $A_{32} \in \Gamma_0(8)$  so that (4.1) gives that

$$\begin{aligned} \left(\frac{\gamma}{\delta}\right) (-1)^{\frac{\alpha-1}{4}} \nu(A_{32})^{-3} &= \left(\frac{32}{\delta}\right) \exp\left(-\frac{\pi i}{4} \left(\alpha - 1 + \frac{\gamma}{32}(\alpha + \delta) - 32\beta\delta\left(\frac{\gamma^2}{32^2} - 1\right) + 3\delta - 3 - \frac{3\gamma\delta}{32}\right)\right) \\ &= \left(\frac{2}{\delta}\right) \exp\left(-\frac{\pi i}{4}(\alpha + 3\delta - 4)\right) \\ &= 1. \end{aligned}$$

Thus

$$\widetilde{\mathcal{R}2}(a, c; 8A\tau) = \left(\frac{\gamma}{\delta}\right) \sqrt{\gamma\tau + \delta} \widetilde{\mathcal{R}2}(a, c; 8\tau).$$

□

Upon noting that  $\Gamma_0(256) \cap \Gamma_0(16c^2) \cap \Gamma_0(32c) \cap \Gamma_1(8) \cap \Gamma_1(2c)$  can be abbreviated to  $\Gamma_0(256t^2) \cap \Gamma_1(8t)$  where  $t = \frac{c}{\gcd(4, c)}$ , we have now proved part (1) of Theorem 2.1. However, it is not  $\widetilde{\mathcal{R}2}(a, c; 8\tau)$  that we will work with for calculations. We instead define

$$F(a, c; \tau) = \frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)} \widetilde{\mathcal{R}2}(a, c; \tau) = f_{4,1}(\tau) \widetilde{\mathcal{R}2}(a, c; \tau),$$

where  $f_{N,\rho}(\tau)$  is the generalized eta function defined for integers  $N, \rho$  with  $N > 1$  and  $N \nmid \rho$  by

$$f_{N,\rho}(\tau) = q^{\frac{(N-2\rho)^2}{8N}} (q^\rho, q^{N-\rho}, q^N; q^N)_\infty.$$

We note that  $f_{N,\rho}(\tau) = f_{N,-\rho}(\tau) = f_{N,\rho+N}(\tau)$ . With this Lemma 2.1 of [4] can be stated as follows, for  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(N)$  we have

$$f_{N,\rho}(A\tau) = (-1)^{\rho\beta + \lfloor \frac{\rho\alpha}{N} \rfloor + \lfloor \frac{\rho}{N} \rfloor} \exp\left(\frac{\pi i \alpha \beta \rho^2}{N}\right) \nu(A_N)^3 \sqrt{\gamma\tau + \delta} f_{N,\rho}(\tau). \quad (4.4)$$

While  $F(a, c; \tau)$  will have at worst linear exponential growth at the cusps, it need not be harmonic. The major advantage of  $F(a, c; \tau)$  over  $\widetilde{\mathcal{R}2}(a, c; \tau)$  is seen in the following proposition.

**Proposition 4.8.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(2c^2) \cap \Gamma_0(4c) \cap \Gamma_1(4) \cap \Gamma_1(2c)$ , then*

$$F(a, c; A\tau) = (\gamma\tau + \delta)F(a, c; \tau).$$

*Proof.* Since  $A \in \Gamma_1(4)$ , by (4.4) we have that

$$f_{4,1}(A\tau) = (-1)^{\beta + \lfloor \frac{\alpha}{4} \rfloor} \exp\left(\frac{\pi i \alpha \beta}{4}\right) \nu(A_4)^3 \sqrt{\gamma\tau + \delta} f_{4,1}(\tau).$$

Since  $\alpha \equiv 1 \pmod{4}$  we have that  $\frac{\alpha-1}{4}$  is an integer and  $\lfloor \frac{\alpha}{4} \rfloor = \frac{\alpha-1}{4}$ . Thus

$$f_{4,1}(A\tau) = (-1)^{\beta + \frac{\alpha-1}{4} + \frac{\beta(\alpha-1)}{4}} \exp\left(\frac{\pi i \beta}{4}\right) \nu(A_4)^3 \sqrt{\gamma\tau + \delta} f_{4,1}(\tau),$$

and so

$$F(a, c; A\tau) = (\gamma\tau + \delta)F(a, c; \tau).$$

□

As noted in the introduction, we should not consider it a coincidence that  $\frac{\eta(2\tau)}{\eta(4\tau)\eta(\tau)}$  is the generating function for partitions without repeated odd parts. Next we define the function

$$H(a, c; \tau) = \frac{\eta(4c^2\tau)\eta(c^2\tau)\eta(2\tau)}{\eta(2c^2\tau)\eta(4\tau)\eta(\tau)} F(a, c; \tau) = \frac{\eta(4c^2\tau)\eta(c^2\tau)}{\eta(2c^2\tau)} \widetilde{\mathcal{R}2}(a, c; \tau).$$

We note that by Theorem 1.64 of [19], when  $c$  is odd,  $\frac{\eta(2c^2\tau)\eta(4\tau)\eta(\tau)}{\eta(4c^2\tau)\eta(c^2\tau)\eta(2\tau)}$  is a modular function on  $\Gamma_0(4c^2)$ . We then have the following.

**Corollary 4.9.** Suppose  $a$  and  $c$  are integers,  $c > 0$ ,  $c$  is odd, and  $c \nmid a$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4c^2) \cap \Gamma_1(4c)$ , then

$$H(a, c; A\tau) = (\gamma\tau + \delta)H(a, c; \tau).$$

**Proposition 4.10.** Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . The non-holomorphic part of  $\widetilde{\mathcal{R}2}(a, c; \tau)$  is

$$\begin{aligned} & \frac{(1 - \zeta_c^a)\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sgn}(n + \tfrac{1}{4}) (\zeta_c^{-2an} - \zeta_c^{2an+a}) q^{-2n^2 - n - \frac{1}{8}} \Gamma(\tfrac{1}{2}, 8\pi(n + \tfrac{1}{4})^2 y) \\ &= \frac{i^{1-c}(1 - \zeta_c^a)\zeta_c^{-a}}{2} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) q^{-\frac{(1+4k-2c)^2}{8}} R(c(1+4k-2c)\tau + \tfrac{1-c}{2}; 4c^2\tau), \end{aligned}$$

where  $y = \operatorname{Im}(\tau)$ .

*Proof.* The non-holomorphic part of  $\widetilde{\mathcal{R}2}(a, c; \tau)$  is

$$-i(1 - \zeta_c^a)\zeta_c^{-a} \left( \exp\left(\frac{\pi i}{4}\right) \int_{-\tau}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)}{\sqrt{-i(w + \tau)}} dw + \exp\left(\frac{-\pi i}{4}\right) \int_{-\tau}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)}{\sqrt{-i(w + \tau)}} dw \right).$$

We find that

$$\begin{aligned} \int_{-\tau}^{i\infty} \frac{g_{\frac{3}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)}{\sqrt{-i(w + \tau)}} dw &= \sum_{n=-\infty}^{\infty} (n + \tfrac{3}{4}) \int_{-\tau}^{i\infty} \frac{e^{\pi i(n + \frac{3}{4})^2 4w + 2\pi i(n + \frac{3}{4})(\frac{1}{2} - \frac{2a}{c})}}{\sqrt{-i(w + \tau)}} dw \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} \zeta_c^{-2an} \zeta_{2c}^{-3a} e^{\frac{3\pi i}{4}}}{4\pi i(n + \frac{3}{4})} \int_{8\pi(n + \frac{3}{4})^2 y}^{\infty} \frac{e^{-t} e^{-4\pi i(n + \frac{3}{4})^2 \tau}}{\sqrt{\frac{t}{4\pi(n + \frac{3}{4})^2}}} dt \\ &= \frac{ie^{\frac{3\pi i}{4}} \zeta_{2c}^{-3a}}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \zeta_c^{-2an} q^{-2n^2 - 3n - \frac{9}{8}} \operatorname{sgn}(n + \tfrac{3}{4}) \Gamma(\tfrac{1}{2}, 8\pi(n + \tfrac{3}{4})^2 y), \end{aligned}$$

where we have used the substitution  $w = -\frac{t}{4\pi i(n + \frac{3}{4})^2} - \tau$ . Similarly we calculate that

$$\int_{-\tau}^{i\infty} \frac{g_{\frac{1}{4}, \frac{1}{2} - \frac{2a}{c}}(4w)}{\sqrt{-i(w + \tau)}} dw = \frac{ie^{\frac{\pi i}{4}} \zeta_{2c}^{-a}}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \zeta_c^{-2an} q^{-2n^2 - n - \frac{1}{8}} \operatorname{sgn}(n + \tfrac{1}{4}) \Gamma(\tfrac{1}{2}, 8\pi(n + \tfrac{1}{4})^2 y).$$

With  $n \mapsto -n - 1$  we find that

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^n \zeta_c^{-2an} q^{-2n^2 - 3n - \frac{9}{8}} \operatorname{sgn}(n + \tfrac{3}{4}) \Gamma(\tfrac{1}{2}, 8\pi(n + \tfrac{3}{4})^2 y) \\ &= \sum_{n=-\infty}^{\infty} (-1)^n \zeta_c^{2an+2a} q^{-2n^2 - n - \frac{1}{8}} \operatorname{sgn}(n + \tfrac{1}{4}) \Gamma(\tfrac{1}{2}, 8\pi(n + \tfrac{1}{4})^2 y). \end{aligned}$$

Thus the nonholomorphic part of  $\widetilde{\mathcal{R}2}(a, c; \tau)$  is

$$\begin{aligned} & \frac{(1 - \zeta_c^a)\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sgn}(n + \tfrac{1}{4}) (\zeta_c^{-2an} - \zeta_c^{2an+a}) q^{-2n^2 - n - \frac{1}{8}} \Gamma(\tfrac{1}{2}, 8\pi(n + \tfrac{1}{4})^2 y) \\ &= \frac{(1 - \zeta_c^a)\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \sum_{n=-\infty}^{\infty} (-1)^{cn} \operatorname{sgn}(cn + k + \tfrac{1}{4}) q^{-2(cn+k+\frac{1}{4})^2} \Gamma(\tfrac{1}{2}, 8\pi(cn + k + \tfrac{1}{4})^2 y) \\ &= \frac{(1 - \zeta_c^a)\zeta_c^{-a}}{2\sqrt{\pi}} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \sum_{n=-\infty}^{\infty} (-1)^{cn} \operatorname{sgn}(n + \tfrac{1+4k}{4c}) q^{-2c^2(n + \frac{1+4k}{4c})^2} \int_{8\pi c^2(n + \frac{1+4k}{4c})^2 y}^{\infty} \frac{e^{-t}}{\sqrt{t}} dt \\ &= -ic(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \sum_{n=-\infty}^{\infty} (-1)^{cn} (n + \tfrac{1+4k}{4c}) \int_{-\tau}^{i\infty} \frac{e^{4\pi i w c^2(n + \frac{1+4k}{4c})^2}}{\sqrt{-i(w + \tau)}} dw \end{aligned}$$

$$\begin{aligned}
&= -ice^{-\frac{\pi i}{4}}(1 - \zeta_c^a)\zeta_c^{-a} \sum_{k=0}^{c-1} (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \int_{-\tau}^{i\infty} \frac{g_{\frac{1+4k-2c}{4c}, \frac{c-1}{2}}(4c^2w)}{\sqrt{-i}(w + \tau)} dw \\
&= -\frac{ie^{-\frac{\pi i}{4}}(1 - \zeta_c^a)\zeta_c^{-a}}{2} \sum_{k=0}^{c-1} (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) \int_{-4c^2\tau}^{i\infty} \frac{g_{\frac{1+4k-2c}{4c}, \frac{c-1}{2}, \frac{c-1}{2} + \frac{1}{2}}(z)}{\sqrt{-i}(z + 4c^2\tau)} dz \\
&= \frac{i^{1-c}(1 - \zeta_c^a)\zeta_c^{-a}}{2} \sum_{k=0}^{c-1} (-1)^k (\zeta_c^{-2ak} - \zeta_c^{2ak+a}) q^{-\frac{(1+4k-2c)^2}{8}} R(c(1+4k-2c)\tau + \frac{1-c}{2}; 4c^2\tau).
\end{aligned}$$

We note that in the second to last equality we have used Theorem 1.16 of [23], which is valid since  $-\frac{1}{2} < \frac{1+4k-2c}{4c} < \frac{1}{2}$  for  $0 \leq k \leq c-1$ .  $\square$

It is worth noting that for  $c$  odd we have  $\zeta_c^{-2ak} - \zeta_c^{2ak+1} = 0$  when  $k \equiv -\frac{1}{4} \pmod{c}$ . When  $c$  is an odd prime, this is exactly when  $-\frac{(1+4k-2c)^2}{8} \equiv 0 \pmod{c}$ . We now see the definitions of  $\mathcal{S}(k, c; \tau)$  and  $\tilde{\mathcal{S}}(k, c; \tau)$  are well motivated. We recall that Proposition 3.3 states that

$$\tilde{\mathcal{S}}(k, c; \tau) = \begin{cases} q^{-\frac{(1+4k-2c)^2}{8}} \tilde{\mu}((2k-c)2c\tau, \frac{c-1}{2} - c\tau; 4c^2\tau) & \text{if } c \text{ is odd,} \\ q^{-\frac{(1+4k-2c)^2}{8}} \tilde{\mu}((1+4k-2c)c\tau, \frac{c-1}{2}; 4c^2\tau) & \text{if } c \text{ is even.} \end{cases}$$

From this it is apparent that we have chosen  $\tilde{\mathcal{S}}(k, c; \tau)$  so that the non-holomorphic part is  $\frac{i}{2}q^{-\frac{(1+4k-2c)^2}{8}}R(c(1+4k-2c)\tau + \frac{1-c}{2}; 4c^2\tau)$ , which matches with the terms of the non-holomorphic part of  $\tilde{\mathcal{R}}2(a, c; \tau)$ .

**Proposition 4.11.** *Suppose  $k$  and  $c$  are integers,  $c > 0$ , and  $c$  is odd. Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4c^2) \cap \Gamma_1(4c)$ , then*

$$\tilde{\mathcal{S}}(k, c; A\tau) = (-1)^{\beta + \frac{(\alpha-1)}{4} + \frac{(\alpha-1)\beta}{4}} \exp(-\frac{\pi i \beta}{4}) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mathcal{S}}(k, c; \tau).$$

*Proof.* By (4.3) we have

$$\begin{aligned}
&\tilde{\mathcal{S}}(k, c; A\tau) \\
&= (-1)^{\frac{c-1}{2}} \exp\left(-\frac{\pi i(1+4k-2c)^2 A\tau}{4}\right) \tilde{\mu}\left(\frac{(2k-c)2c(\alpha\tau + \beta)}{(\gamma\tau + \delta)}, -\frac{c(\alpha\tau + \beta)}{(\gamma\tau + \delta)}, A_{4c^2}(4c^2\tau)\right) \\
&= (-1)^{\frac{c-1}{2}} \exp\left(-\frac{\pi i(1+4k-2c)^2 A\tau}{4}\right) \nu(A_{4c^2})^{-3} \exp\left(\left(-\frac{\pi i \gamma}{4c^2(\gamma\tau + \delta)}((2k-c)2c(\alpha\tau + \beta) + c(\alpha\tau + \beta))^2\right)\right. \\
&\quad \cdot \sqrt{\gamma\tau + \delta} \tilde{\mu}((2k-c)2c(\alpha\tau + \beta), -c(\alpha\tau + \beta), 4c^2\tau) \\
&= (-1)^{\frac{c-1}{2} + \beta} \exp\left(-\frac{\pi i(1+4k-2c)^2 \alpha(\alpha\tau + \beta)}{4}\right) \nu(A_{4c^2})^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mu}((2k-c)2c\alpha\tau, -c\alpha\tau, 4c^2\tau).
\end{aligned}$$

Now we use that  $\alpha \equiv 1 \pmod{4c}$  so that we may write  $\alpha = 1 + \frac{(\alpha-1)}{4c}4c$  and use (4.2) to get that

$$\tilde{\mathcal{S}}(k, c; \tau) = (-1)^{\beta + \frac{(\alpha-1)}{4} + \frac{(\alpha-1)\beta}{4}} \exp(-\frac{\pi i \beta}{4}) \nu(A_{4c^2})^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mathcal{S}}(k, c; \tau).$$

Lastly we note that by Theorem 1.64 of [19], for  $c$  odd we have  $\frac{\eta(4c^2\tau)^3}{\eta(4\tau)^3}$  is a modular function on  $\Gamma_0(4c^2)$ , so that  $\nu(A_{4c^2})^3 = \nu(A_4)^3$ .  $\square$

**Proposition 4.12.** *Suppose  $k$  and  $c$  are integers,  $c > 0$ , and  $c$  is even. Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(8c^2) \cap \Gamma_1(4c)$ , then*

$$\tilde{\mathcal{S}}(k, c; A\tau) = (-1)^\beta \exp(-\frac{\pi i \beta}{4}) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} \tilde{\mathcal{S}}(k, c; \tau).$$

*We note that this is the same transformation formula as for  $\tilde{\mathcal{R}}2(a, c; \tau)$ .*

*Proof.* By (4.3) we have

$$\begin{aligned}
&\tilde{\mathcal{S}}(k, c; A\tau) \\
&= \exp\left(-\frac{\pi i(1+4k-2c)^2 A\tau}{4}\right) \tilde{\mu}\left((1+4k-2c)cA\tau, \frac{c-1}{2}; A_{4c^2}(4c^2\tau)\right) \\
&= \nu(A_{4c^2})^{-3} \exp\left(-\frac{\pi i(1+4k-2c)^2(\alpha^2\tau + \alpha\beta)}{4}\right) \exp\left(-\frac{\pi i \gamma}{4c^2}\left(-(1+4k-2c)c(c-1)(\alpha\tau + \beta) + \frac{(c-1)^2}{4}(\gamma\tau + \delta)\right)\right)
\end{aligned}$$

$$\cdot \sqrt{\gamma\tau + \delta} \tilde{\mu} \left( (1+4k-2c)c(\alpha\tau + \beta), \frac{c-1}{2}(\gamma\tau + \delta); 4c^2\tau \right).$$

Using that  $\alpha \equiv 1 \pmod{4c}$ ,  $\delta \equiv 1 \pmod{4}$ , and  $\gamma \equiv 0 \pmod{8c^2}$  with (4.2), we find that

$$\begin{aligned} & \tilde{\mu} \left( (1+4k-2c)c(\alpha\tau + \beta), \frac{c-1}{2}(\gamma\tau + \delta); 4c^2\tau \right) \\ &= \exp \left( \pi i \tau \left( (1+4k-2c)\frac{(\alpha-1)}{2} - \frac{(c-1)\gamma}{4c} \right) \left( (1+4k-2c)\frac{(\alpha+1)}{2} - \frac{(c-1)\gamma}{4c} \right) \right) \tilde{\mu} \left( (1+4k-2c)c\tau, \frac{c-1}{2}; 4c^2\tau \right). \end{aligned}$$

With this we have that

$$\begin{aligned} & \tilde{\mathcal{S}}(k, c; A\tau) \\ &= \nu(A_{4c^2})^{-3} \exp \left( -\frac{\pi i \tau}{4} \left( (1+4k-2c)^2 \alpha^2 + \frac{\gamma}{c^2} \left( -(1+4k-2c)c(c-1)\alpha + \frac{(c-1)^2 \gamma}{4} \right) \right. \right. \\ & \quad \left. \left. - \left( (1+4k-2c)(\alpha-1) - \frac{(c-1)\gamma}{2c} \right) \left( (1+4k-2c)(\alpha+1) - \frac{(c-1)\gamma}{2c} \right) \right) \right) \\ & \quad \cdot \exp \left( -\frac{\pi i \gamma (c-1)^2 \delta}{16c^2} - \frac{\pi i (1+4k-2c)^2 \alpha \beta}{4} \right) \sqrt{\gamma\tau + \delta} \tilde{\mu} \left( (1+4k-2c)c\tau, \frac{c-1}{2}; 4c^2\tau \right) \\ &= \nu(A_{4c^2})^{-3} \exp \left( -\frac{\pi i \gamma (c-1)^2 \delta}{16c^2} - \frac{\pi i \alpha \beta}{4} \right) \sqrt{\gamma\tau + \delta} \tilde{\mathcal{S}}(k, c; \tau). \end{aligned}$$

However, from  $\alpha \equiv 1 \pmod{4c}$  we have that  $\alpha \equiv 1 \pmod{8}$ . Along with  $\delta \equiv 1 \pmod{4}$  this gives that

$$\exp \left( -\frac{\pi i \gamma (c-1)^2 \delta}{16c^2} - \frac{\pi i \alpha \beta}{4} \right) = i^{-\frac{\gamma}{8c^2}} \exp \left( -\frac{\pi i \beta}{4} \right).$$

It then only remains to verify that

$$\nu(A_{4c^2})^{-3} i^{-\frac{\gamma}{8c^2}} = (-1)^\beta \nu(A_4)^{-3}.$$

Using (4.1) along with the facts that  $\alpha \equiv \delta \equiv 1 \pmod{8}$  and  $\gamma \equiv 0 \pmod{32}$  we find that

$$\begin{aligned} & (-1)^\beta i^{-\frac{\gamma}{8c^2}} \nu(A_{4c^2})^{-3} \nu(A_4)^3 \\ &= \left( \frac{\gamma/4c^2}{\delta} \right)^3 \left( \frac{\gamma/4}{\delta} \right)^3 \exp \left( \frac{\pi i}{4} \left( 4\beta - \frac{\gamma}{4c^2} + (\alpha + \delta)\frac{\gamma}{4} - 4\beta\delta \left( \frac{\gamma^2}{16} - 1 \right) + 3\delta - 3 - \frac{3\gamma\delta}{4} - (\alpha + \delta)\frac{\gamma}{4c^2} \right. \right. \\ & \quad \left. \left. + 4c^2\beta\delta \left( \frac{\gamma^2}{16c^4} - 1 \right) - 3\delta + 3 + \frac{3\gamma\delta}{4c^2} \right) \right) \\ &= \exp \left( \frac{\pi i}{2} \frac{\gamma}{8c^2} (-1 - \alpha - \delta + \beta\delta\gamma + 3\delta) \right) \\ &= 1. \end{aligned}$$

□

**Proposition 4.13.** Suppose  $k$  and  $c$  are integers,  $c > 0$ , and  $c$  is odd. Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ , then

$$\tilde{\mathcal{S}}(k, c; \frac{1}{4c^2} A\tau) = (-1)^{\frac{c-1}{2}} \nu(A)^{-3} \exp \left( -\frac{\pi i \alpha \beta (1+4k-2c)^2}{16c^2} \right) \sqrt{\gamma\tau + \delta} q^{-\frac{\alpha^2(1+4k-2c)^2}{32c^2}} \tilde{\mu} \left( \frac{(2k-c)(\alpha\tau + \beta)}{2c}, -\frac{(\alpha\tau + \beta)}{4c}; \tau \right).$$

*Proof.* By (4.3) we have that

$$\begin{aligned} & \tilde{\mathcal{S}}(k, c; \frac{1}{4c^2} A\tau) \\ &= (-1)^{\frac{c-1}{2}} \exp \left( -\frac{\pi i (1+4k-2c)^2 A\tau}{16c^2} \right) \tilde{\mu} \left( \frac{(2k-c)2cA\tau}{4c^2}, -\frac{cA\tau}{4c^2}; A\tau \right) \\ &= (-1)^{\frac{c-1}{2}} \exp \left( -\frac{\pi i \alpha \beta (1+4k-2c)^2}{16c^2} \right) \nu(A)^{-3} \sqrt{\gamma\tau + \delta} q^{-\frac{\alpha^2(1+4k-2c)^2}{32c^2}} \tilde{\mu} \left( \frac{(2k-c)(\alpha\tau + \beta)}{2c}, -\frac{(\alpha\tau + \beta)}{4c}; \tau \right). \end{aligned}$$

□

**Proposition 4.14.** Suppose  $k$  and  $c$  are integers,  $c > 0$ , and  $c$  is even. Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ , then

$$\begin{aligned} & \tilde{\mathcal{S}}(k, c; \frac{1}{4c^2} A\tau) = \nu(A)^{-3} \exp \left( -\frac{\pi i \alpha \beta (1+4k-2c)^2}{16c^2} - \frac{\pi i \gamma (c-1)}{4c} (-(1+4k-2c)\beta + c(c-1)\delta) \right) \\ & \quad \cdot \sqrt{\gamma\tau + \delta} q^{-\frac{(2c\gamma(c-1) - \alpha(1+4k-2c))^2}{32c^2}} \tilde{\mu} \left( \frac{(1+4k-2c)(\alpha\tau + \beta)}{4c}, \frac{(c-1)(\gamma\tau + \delta)}{2}; \tau \right). \end{aligned}$$



*Proof.* By (4.3) we have that

$$\begin{aligned}
& \tilde{\mathcal{S}}(k, c; \frac{1}{4c^2}A\tau) \\
&= \exp\left(-\frac{\pi i(1+4k-2c)^2}{16c^2}A\tau\right) \tilde{\mu}\left(\frac{(1+4k-2c)}{4c}A\tau, \frac{c-1}{2}; A\tau\right) \\
&= \nu(A)^{-3} \exp\left(-\frac{\pi i(1+4k-2c)^2\alpha\beta}{16c^2} - \frac{\pi i\gamma(c-1)}{4c}(-(1+4k-2c)\beta + c(c-1)\delta)\right) q^{-\frac{(2c\gamma(c-1)-\alpha(1+4k-2c))^2}{32c^2}} \\
&\quad \cdot \sqrt{\gamma\tau + \delta} \tilde{\mu}\left(\frac{(1+4k-2c)(\alpha\tau + \beta)}{4c}, \frac{(c-1)(\gamma\tau + \delta)}{2}; \tau\right).
\end{aligned}$$

□

**Proposition 4.15.** *Suppose  $k$  and  $c$  are integers and  $c > 0$ . Then  $\tilde{\mathcal{S}}(k, c; \tau)$  has at worst linear exponential growth at the cusps and is annihilated by  $\Delta_{\frac{1}{2}}$ .*

*Proof.* We first verify that  $\tilde{\mathcal{S}}(k, c; \tau)$  has at worst linear exponential growth at the cusps. For this, suppose  $\alpha/\gamma$  is in reduced form. We then take  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ . We set  $g = \gcd(4c^2, \gamma)$ ,  $x = 4c^2\alpha/g$ , and  $u = \gamma/g$ . Since  $\gcd(x, u) = 1$  we can take  $L = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , and set

$$B = L^{-1} \begin{pmatrix} 4c^2\alpha & 4c^2\beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} g & 4c^2\beta v - \delta y \\ 0 & 4c^2/g \end{pmatrix}.$$

We note that  $j(L, B\tau) = \frac{g}{4c^2}(\gamma\tau + \delta)$ .

When  $c$  is odd, we have with Proposition 4.13 that

$$\begin{aligned}
(\gamma\tau + \delta)^{-\frac{1}{2}} \tilde{\mathcal{S}}(k, c; A\tau) &= (\gamma\tau + \delta)^{-\frac{1}{2}} \tilde{\mathcal{S}}(k, c; \frac{1}{4c^2}LB\tau) \\
&= \frac{\sqrt{g}}{2c}(-1)^{\frac{c-1}{2}} \nu(L)^{-3} \exp\left(-\frac{\pi ixy(1+4k-2c)^2}{16c^2}\right) \exp\left(-\frac{\pi ix^2(1+4k-2c)^2}{16c^2}B\tau\right) \\
&\quad \cdot \tilde{\mu}\left(\frac{(2k-c)x}{2c}B\tau + \frac{(2k-c)y}{2c}, -\frac{x}{4c}B\tau - \frac{y}{4c}; B\tau\right).
\end{aligned} \tag{4.5}$$

When  $c$  is even we instead use Proposition 4.14 to find that

$$\begin{aligned}
& (\gamma\tau + \delta)^{-\frac{1}{2}} \tilde{\mathcal{S}}(k, c; A\tau) \\
&= (\gamma\tau + \delta)^{-\frac{1}{2}} \tilde{\mathcal{S}}(k, c; \frac{1}{4c^2}LB\tau) \\
&= \frac{\sqrt{g}}{2} \nu(L)^{-3} \exp\left(-\frac{\pi ixy(1+4k-2c)^2}{16c^2} - \frac{\pi iu(c-1)}{4c}(-(1+4k-2c)y + c(c-1)v)\right) \\
&\quad \cdot \exp\left(-\frac{\pi i(2cu(c-1)-x(1+4k-2c))^2}{16c^2}B\tau\right) \tilde{\mu}\left(\frac{(1+4k-2c)x}{4c}B\tau + \frac{(1+4k-2c)y}{4c}, \frac{(c-1)u}{2}B\tau + \frac{(c-1)v}{2}; B\tau\right).
\end{aligned} \tag{4.6}$$

We note that  $B\tau = \frac{g^2}{4c^2}\tau + \frac{g(4c^2\beta v - \delta y)}{4c^2}$ , and so  $e^{\pi iB\tau} = \epsilon q^{\frac{g^2}{8c^2}}$ , where  $|\epsilon| = 1$ . From this we see  $(\gamma\tau + \delta)^{-\frac{1}{2}} \tilde{\mathcal{S}}(k, c; A\tau)$  has at worst linear exponential growth as  $\tau \rightarrow i\infty$  by the definition of  $\tilde{\mu}$ .

Next we verify that  $\tilde{\mathcal{S}}(k, c; \tau)$  is annihilated by  $\Delta_{\frac{1}{2}}$ . For this we need only verify that

$$\sqrt{y} \frac{\partial}{\partial \bar{\tau}} q^{-\frac{(1+4k-2c)^2}{8}} R((1+4k-c)c\tau + \frac{(1-c)}{2}; 4c^2\tau)$$

is holomorphic in  $\bar{\tau}$ , where  $y = \text{Im}(\tau)$ . However, using Lemma 1.8 of [23] we find that

$$\sqrt{y} \frac{\partial}{\partial \bar{\tau}} q^{-\frac{(1+4k-2c)^2}{8}} R((1+4k-c)c\tau + \frac{(1-c)}{2}; 4c^2\tau) = -i\sqrt{2}c \exp\left(-\frac{\pi i}{4}(1+4k-2c)^2\bar{\tau}\right) A(\bar{\tau}),$$

where  $A(\bar{\tau})$  is a series defining a function holomorphic in  $\bar{\tau}$ . Thus  $\Delta_{\frac{1}{2}}\tilde{\mathcal{S}}(k, c; \tau) = 0$ . □

We can multiply  $\tilde{\mathcal{S}}(k, c; \tau)$  by  $\frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)}$  and  $\frac{\eta(4c^2\tau)\eta(c^2\tau)}{\eta(2c^2\tau)}$  and then deduce the same transformation formulas as we did for  $\widetilde{\mathcal{R}2}(a, c; \tau)$ . Taking the correct difference of these terms will cancel the non-holomorphic parts and so the result will be a modular form. This establishes parts (2), (3), and (4) of Theorem 2.1. To accommodate the generalized eta-quotients in the dissection of  $R2(\zeta_7; \tau)$  in Theorem 2.2, we need the following results. We recall that  $f_{N,\rho}(\tau) = q^{\frac{(N-2\rho)^2}{8N}} (q^\rho, q^{N-\rho}, q^N; q^N)_\infty$ .

**Proposition 4.16.** *Suppose  $c > 0$  is an integer and*

$$f(\tau) = \eta(4c^2\tau)^{r_0} \prod_{k=1}^{2c} f_{4c^2,ck}(\tau)^{r_k}.$$

*Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4c^2) \cap \Gamma_1(4c)$ , then*

$$f(A\tau) = (-1)^{(c\beta + \frac{(\alpha-1)}{4c} + \frac{\beta(\alpha-1)}{4})S + \beta T} \exp\left(\frac{\pi i \beta S}{4}\right) \nu(A_{4c^2})^{r_0+3R} (\gamma\tau + \delta)^{\frac{r_0+R}{2}} f(\tau),$$

*where*

$$R = \sum_{i=1}^{2c} r_i, \quad S = \sum_{\substack{i=1, \\ i \equiv 1 \pmod{2}}}^{2c} r_i, \quad T = \sum_{\substack{i=1, \\ i \equiv 2 \pmod{4}}}^{2c} r_i.$$

*Proof.* By (4.4) we have

$$f_{4c^2,ck}(A\tau) = (-1)^{kc\beta + \lfloor \frac{kc\alpha}{4c^2} \rfloor + \lfloor \frac{kc}{4c^2} \rfloor} \exp\left(\frac{\pi i \alpha \beta (kc)^2}{4c^2}\right) \nu(A_{4c^2})^3 \sqrt{\gamma\tau + \delta} f_{4c^2,ck}(\tau).$$

Since  $\alpha \equiv 1 \pmod{4c}$  and  $1 \leq k \leq 2c$ , we can write  $\alpha = 1 + \frac{(\alpha-1)}{4c}4c$  to deduce that

$$\left\lfloor \frac{kc\alpha}{4c^2} \right\rfloor = \frac{(\alpha-1)k}{4c}.$$

Thus

$$(-1)^{kc\beta + \lfloor \frac{kc\alpha}{4c^2} \rfloor + \lfloor \frac{kc}{4c^2} \rfloor} = (-1)^{k(c\beta + \frac{(\alpha-1)}{4c})}.$$

Next we have that

$$\exp\left(\frac{\pi i \alpha \beta (kc)^2}{4c^2}\right) = (-1)^{k \frac{\beta(\alpha-1)}{4}} \exp\left(\frac{\pi i \beta k^2}{4}\right).$$

So in fact

$$f_{4c^2,ck}(A\tau) = (-1)^{k(c\beta + \frac{(\alpha-1)}{4c} + \frac{\beta(\alpha-1)}{4})} \exp\left(\frac{\pi i \beta k^2}{4}\right) \nu(A_{4c^2})^3 \sqrt{\gamma\tau + \delta} f_{4c^2,ck}(\tau).$$

Thus

$$\begin{aligned} f(A\tau) &= \left( \prod_{k=1}^{2c} (-1)^{r_k k(c\beta + \frac{(\alpha-1)}{4c} + \frac{\beta(\alpha-1)}{4})} \exp\left(\frac{\pi i \beta k^2 r_k}{4}\right) \right) \nu(A_{4c^2})^{r_0+3R} (\gamma\tau + \delta)^{\frac{r_0+R}{2}} f(\tau) \\ &= (-1)^{(c\beta + \frac{(\alpha-1)}{4c} + \frac{\beta(\alpha-1)}{4})S} (-1)^{\beta T} \exp\left(\frac{\pi i \beta S}{4}\right) \nu(A_{4c^2})^{r_0+3R} (\gamma\tau + \delta)^{\frac{r_0+R}{2}} f(\tau). \end{aligned}$$

□

**Corollary 4.17.** *Suppose  $c > 0$  is an odd integer,*

$$f(\tau) = \eta(4c^2\tau)^{r_0} \prod_{k=1}^{2c} f_{4c^2,ck}(\tau)^{r_k},$$

*and*

$$R = \sum_{i=1}^{2c} r_i, \quad S = \sum_{\substack{i=1, \\ i \equiv 1 \pmod{2}}}^{2c} r_i, \quad T = \sum_{\substack{i=1, \\ i \equiv 2 \pmod{4}}}^{2c} r_i.$$

*Suppose that  $r_0 + R = 1$ ,  $R \equiv -2 \pmod{12}$ ,  $S \equiv -1 \pmod{8}$ , and  $T \equiv 0 \pmod{2}$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4c^2) \cap \Gamma_1(4c)$ , then*

$$f(A\tau) = (-1)^{\beta + \frac{(\alpha-1)}{4} + \frac{\beta(\alpha-1)}{4}} \exp\left(-\frac{\pi i \beta}{4}\right) \nu(A_4)^{-3} \sqrt{\gamma\tau + \delta} f(\tau).$$

*Furthermore if*

$$g(\tau) = \frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)} f(\tau), \quad h(\tau) = \frac{\eta(4c^2\tau)\eta(c^2\tau)}{\eta(2c^2\tau)} f(\tau),$$

then  $g(\tau)$  and  $h(\tau)$  are a weight 1 weakly holomorphic modular forms on  $\Gamma_0(4c^2) \cap \Gamma_1(4c)$ .

*Proof.* The only simplification that does not follow immediately from Proposition 4.16 and the conditions on  $R$ ,  $S$ , and  $T$  is that  $\nu(A_{4c^2})^{r_0+3R} = \nu(A_4)^{-3}$ . However this follows from the fact that since  $c$  is odd, we have that  $\frac{\eta(4c^2\tau)^3}{\eta(4\tau)^3}$  is a modular function on  $\Gamma_0(4c^2)$  by Theorem 1.64 of [19]. From this we have that  $\nu(A_{4c^2})^3 = \nu(A_4)^3$ . But we also know  $\nu(A_{4c^2})^3$  is a  $24^{th}$  root of unity and  $r_0 + 3R = 1 + 2R \equiv -3 \pmod{24}$ , so that  $\nu(A_{4c^2})^{r_0+3R} = \nu(A_{4c^2})^{-3} = \nu(A_4)^3$ .  $\square$

One can easily deduce an analogous result for even  $c$ , however we omit this result as we would not make use of it.

## 5. ORDERS AT CUSPS

For a real number  $x$ , we let  $\lfloor x \rfloor$  denote the greatest integer less than or equal to  $x$  and  $\{x\}$  the fractional part of  $x$ . That is,  $x = \lfloor x \rfloor + \{x\}$ ,  $\lfloor x \rfloor \in \mathbb{Z}$ , and  $0 \leq \{x\} < 1$ . We use Corollary 6.2 from [13].

**Corollary 5.1.** *If  $f(\tau) = q^\alpha \tilde{\mu}(u\tau + v, w\tau + x; \tau)$  is a harmonic weak Maass form, with  $u, v, w, x \in \mathbb{R}$ , then the lowest power of  $q$  appearing in the expansion of the holomorphic part of  $f(\tau)$  is at least  $\alpha + \tilde{\nu}(u, w)$ , where*

$$\begin{aligned} \tilde{\nu}(u, w) &= \frac{1}{2} (\lfloor u \rfloor - \lfloor w \rfloor)^2 + (\lfloor u \rfloor - \lfloor w \rfloor) (\{u\} - \{w\}) + k(u, w), \\ k(u, w) &= \begin{cases} \nu(\{u\}, \{w\}) & \text{if } \{u\} - \{w\} \neq \pm \frac{1}{2}, \\ \min\left(\frac{1}{8}, \nu(\{u\}, \{w\})\right) & \text{if } \{u\} - \{w\} = \pm \frac{1}{2}, \end{cases} \\ \nu(u, w) &= \begin{cases} \frac{u+w}{2} - \frac{1}{8} & \text{if } u + w \leq 1, \\ \frac{7}{8} - \frac{u+w}{2} & \text{if } u + w > 1. \end{cases} \end{aligned}$$

We recall for a modular form  $f$  on some congruence subgroup  $\Gamma$ , the invariant order at  $i\infty$  is the least power of  $q$  appearing in the  $q$ -expansion at  $i\infty$ . That is, if

$$f(\tau) = \sum_{m=m_0}^{\infty} a(m) \exp(2\pi i \tau m / N),$$

and  $a(m_0) \neq 0$ , then the invariant order is  $m_0/N$ . For a modular form, this is always a finite number. For a harmonic weak Maass form, we cannot take such an expansion, however we can do so for the holomorphic part. If  $f$  is a modular form of weight  $k$ ,  $\gcd(\alpha, \gamma) = 1$ , and  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , then the invariant order of  $f$  at the cusp  $\frac{\alpha}{\gamma}$  is the invariant order at  $i\infty$  of  $(A : \tau)^{-k} f(A\tau)$ . In the same fashion, if  $f$  is a harmonic weak Maass form, then the invariant order of the holomorphic part of  $f$  at the cusp  $\frac{\alpha}{\gamma}$  is the invariant order at  $i\infty$  of the holomorphic part of  $(A : \tau)^{-k} f(A\tau)$ . This value is independent of the choice of  $A$ .

**Proposition 5.2.** *Suppose  $a$  and  $c$  are integers,  $c > 0$ , and  $c \nmid 2a$ . Suppose  $\alpha$  and  $\gamma$  are integers and  $\gcd(\alpha, \gamma) = 1$ . Then the invariant order of the holomorphic part of  $\mathcal{R}2(a, c; \tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is at least*

$$\frac{g^2}{4} \left( -\frac{x^2}{32} - \frac{2a^2u}{c^2} + \min \left( -\frac{axu}{2c} + \tilde{\nu} \left( \frac{2au}{c}, -\frac{x}{4} \right), \frac{axu}{2c} + \tilde{\nu} \left( \frac{2au}{c}, \frac{x}{4} \right) \right) \right),$$

where  $g = \gcd(4, \gamma)$ ,  $x = 4\alpha/g$ , and  $u = \gamma/g$ .

*Proof.* As in the proof of Proposition 4.6 we can take  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ ,  $L = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , and set

$$B = L^{-1} \begin{pmatrix} 4\alpha & 4\beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} g & 4\beta v - \delta y \\ 0 & 4/g \end{pmatrix}.$$

Then we have

$$\begin{aligned} & (\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{\mathcal{R}2}(a, c; A\tau) \\ &= i(1 - \zeta_c^a) \left( \zeta_c^{-a} (\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{M}_-(a, c; 4A\tau) + (\gamma\tau + \delta)^{-\frac{1}{2}} \widetilde{M}_+(a, c; 4A\tau) \right) \\ &= i(1 - \zeta_c^a) \frac{\sqrt{g}}{2} \zeta_c^{-2a^2uv} \exp\left(-\frac{\pi ixy}{16}\right) \nu(L)^{-3} \end{aligned}$$

$$\begin{aligned} & \cdot \left( \zeta_c^{-a} \zeta_{2c}^{-ayu} \exp \left( -\pi i \left( \frac{x^2}{16} + \frac{4a^2u}{c^2} + \frac{axu}{c} \right) B\tau \right) \tilde{\mu} \left( \frac{2au}{c} B\tau + \frac{2av}{c}, -\frac{x}{4} B\tau - \frac{y}{4}; B\tau \right) \right. \\ & \left. + \zeta_{2c}^{ayu} \exp \left( -\pi i \left( \frac{x^2}{16} + \frac{4a^2u}{c^2} - \frac{axu}{c} \right) B\tau \right) \tilde{\mu} \left( \frac{2au}{c} B\tau + \frac{2av}{c}, \frac{x}{4} B\tau + \frac{y}{4}; B\tau \right) \right). \end{aligned}$$

Noting  $B\tau = \frac{g^2}{4}\tau + \frac{g(4\beta v - \delta y)}{4}$ , Corollary 5.1 tells us that the invariant order at  $i\infty$  of  $(c\tau + d)^{-\frac{1}{2}} \widetilde{\mathcal{R}2}(a, c; A\tau)$  is at least

$$\begin{aligned} & \min \left( -\frac{g^2}{8} \left( \frac{x^2}{16} + \frac{4a^2u}{c^2} + \frac{axu}{c} \right) + \frac{g^2}{4} \tilde{\nu} \left( \frac{2au}{c}, -\frac{x}{4} \right), -\frac{g^2}{8} \left( \frac{x^2}{16} + \frac{4a^2u}{c^2} - \frac{axu}{c} \right) + \frac{g^2}{4} \tilde{\nu} \left( \frac{2au}{c}, \frac{x}{4} \right) \right) \\ & = \frac{g^2}{4} \left( -\frac{x^2}{32} - \frac{2a^2u}{c^2} + \min \left( -\frac{axu}{2c} + \tilde{\nu} \left( \frac{2au}{c}, -\frac{x}{4} \right), \frac{axu}{2c} + \tilde{\nu} \left( \frac{2au}{c}, \frac{x}{4} \right) \right) \right). \end{aligned}$$

□

**Proposition 5.3.** *Suppose  $k$  and  $c$  are integers and  $c > 0$ . Suppose  $\alpha$  and  $\gamma$  are integers and  $\gcd(\alpha, \gamma) = 1$ . Let  $g = \gcd(4c^2, \gamma)$ ,  $x = 4c^2\alpha/g$ , and  $u = \gamma/g$ . If  $c$  is odd, then the invariant order of the holomorphic part of  $\tilde{\mathcal{S}}(k, c; \tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is at least*

$$-\frac{x^2(1+4k-2c)^2g^2}{128c^4} + \frac{g^2}{4c^2} \tilde{\nu} \left( \frac{(2k-c)x}{2c}, -\frac{x}{4c} \right).$$

If  $c$  is even, then the invariant order of the holomorphic part of  $\tilde{\mathcal{S}}(k, c; \tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is at least

$$-\frac{(2cu(c-1) - x(1+4k-2c))^2g^2}{128c^4} + \frac{g^2}{4c^2} \tilde{\nu} \left( \frac{(1+4k-2c)x}{4c}, \frac{(c-1)u}{2} \right).$$

*Proof.* We take  $A$ ,  $L$ , and  $B$  as in the proof of Proposition 4.15. We recall that  $B\tau = \frac{g^2}{4c^2}\tau + \frac{g(4c^2\beta v - \delta y)}{4c^2}$  and  $e^{\pi i B\tau} = \epsilon q^{\frac{g^2}{8c^2}}$ , where  $|\epsilon| = 1$ . When  $c$  is odd, we use (4.5), which is

$$\begin{aligned} (\gamma\tau + \delta)^{-\frac{1}{2}} \tilde{\mathcal{S}}(k, c; A\tau) &= \frac{\sqrt{g}}{2c} (-1)^{\frac{c-1}{2}} \nu(L)^{-3} \exp \left( -\frac{\pi i x y (1+4k-2c)^2}{16c^2} \right) \exp \left( -\frac{\pi i x^2 (1+4k-2c)^2}{16c^2} B\tau \right) \\ &\quad \cdot \tilde{\mu} \left( \frac{(2k-c)x}{2c} B\tau + \frac{(2k-c)y}{2c}, -\frac{x}{4c} B\tau - \frac{y}{4c}; B\tau \right). \end{aligned}$$

With Corollary 5.1 a lower bound for the order at  $\frac{\alpha}{\gamma}$  is given by

$$-\frac{x^2(1+4k-2c)^2g^2}{128c^4} + \frac{g^2}{4c^2} \tilde{\nu} \left( \frac{(2k-c)x}{2c}, -\frac{x}{4c} \right).$$

When  $c$  is even, we use (4.6), which is

$$\begin{aligned} & (\gamma\tau + \delta)^{-\frac{1}{2}} \tilde{\mathcal{S}}(k, c; A\tau) \\ &= \frac{\sqrt{g}}{2} \nu(L)^{-3} \exp \left( -\frac{\pi i x y (1+4k-2c)^2}{16c^2} - \frac{\pi i u (c-1)}{4c} (-(1+4k-2c)y + c(c-1)v) \right) \\ &\quad \cdot \exp \left( -\frac{\pi i (2cu(c-1) - x(1+4k-2c))^2}{16c^2} B\tau \right) \tilde{\mu} \left( \frac{(1+4k-2c)x}{4c} B\tau + \frac{(1+4k-2c)y}{4c}, \frac{(c-1)u}{2} B\tau + \frac{(c-1)v}{2}; B\tau \right). \end{aligned}$$

With Corollary 5.1 a lower bound for the order at  $\frac{\alpha}{\gamma}$  is given by

$$-\frac{(2cu(c-1) - x(1+4k-2c))^2g^2}{128c^4} + \frac{g^2}{4c^2} \tilde{\nu} \left( \frac{(1+4k-2c)x}{4c}, \frac{(c-1)u}{2} \right).$$

□

The following is Lemma 3.2 of [4].

**Proposition 5.4.** *Suppose  $\alpha$  and  $\gamma$  are integers and  $\gcd(\alpha, \gamma) = 1$ . Then the invariant order of  $f_{N,\rho}(\tau)$  at the cusp  $\frac{\alpha}{\gamma}$  is  $\frac{\gcd(N, \gamma)^2}{2N} \left( \left\{ \frac{\alpha\rho}{\gcd(N, \gamma)} \right\} - \frac{1}{2} \right)^2$ .*

## 6. PROOF OF THEOREM 2.2

To begin we give the definitions of the remaining  $R2_i(q)$  from the statement of Theorem 2.2. We let

$$\begin{aligned}
 R2_1(q) = & \frac{A(-8,1,-3)q^{-3}J_0J_7J_8J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(10,-7,2)q^{-3}J_0J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{13}J_{14}^2} + \frac{A(0,-1,0)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(-5,11,0)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}^2} + \frac{A(4,10,3)q^{-3}J_0J_6J_8J_9^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(0,-1,0)q^{-3}J_0J_6J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(1,2,1)q^{-3}J_0J_6J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(-1,1,0)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{12}J_{13}J_{14}^2} + \frac{A(8,-15,1)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-2,-17,-1)q^{-3}J_0J_6J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(-5,19,1)q^{-3}J_0J_6J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(4,-11,0)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{12}J_{13}J_{14}} + \frac{A(1,9,2)q^{-3}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{14}} + \frac{A(-11,8,-4)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{13}J_{14}} + \frac{A(-3,-2,-2)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}^2} \\
 & + \frac{A(5,-11,0)q^{-3}J_0J_6J_7J_8J_{14}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}^2} + \frac{A(4,2,3)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(3,-11,0)q^{-2}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(3,3,1)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-3,-3,-1)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}^2} + \frac{A(3,2,1)q^{-3}J_0J_6J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-3,-2,-2)q^{-3}J_0J_6J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{13}J_{14}} \\
 & + \frac{A(-3,11,0)q^{-2}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1,0,-1)q^{-2}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(1,-18,-3)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(-5,11,0)q^{-3}J_0J_6^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-4,16,1)q^{-3}J_0J_6^2J_8J_{11}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{12}J_{13}J_{14}^2} + \frac{A(-1,5,0)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{12}J_{13}J_{14}} + \frac{A(4,-16,-1)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(-2,-1,-1)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{13}J_{14}} + \frac{A(3,3,1)q^{-3}J_0J_6^2J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} + \frac{A(0,-1,1)q^{-3}J_0J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{12}J_{13}J_{14}^2} + \frac{A(-2,-4,-1)q^{-3}J_0J_6^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(3,2,2)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}^2} + \frac{A(-1,3,-1)q^{-3}J_0J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(3,3,1)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(1,0,1)q^{-2}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}^2} \\
 & + \frac{A(-1,0,-1)q^{-3}J_0J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_7J_{11}J_{12}J_{13}J_{14}} + \frac{A(0,-1,0)q^{-3}J_0J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5J_7J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1,-1,-1)J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_{10}J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(2,1,1)qJ_0J_5J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_7J_9J_{10}J_{11}J_{12}J_{13}J_{14}^2}, \\
 R2_2(q) = & \frac{A(5,-1,5)q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(0,2,2)q^{-3}J_0J_7^2J_8J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(-12,-6,-16)q^{-3}J_0J_7^2J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(1,-4,1)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(13,8,14)q^{-3}J_0J_7^2J_8^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(3,0,6)q^{-3}J_0J_6J_8J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(2,4,1)q^{-3}J_0J_6J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(3,5,4)q^{-3}J_0J_6J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(-7,-2,-11)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(4,-5,9)q^{-3}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(-11,1,-13)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{13}J_{14}^2} \\
 & + \frac{A(-9,-5,-9)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(0,-1,1)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(2,3,1)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{12}J_{13}J_{14}} + \frac{A(-6,-4,-5)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{14}^2} \\
 & + \frac{A(-3,-2,-2)q^{-3}J_0J_6J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{13}J_{14}} + \frac{A(10,9,7)q^{-3}J_0J_6J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(3,1,5)q^{-3}J_0J_6J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}^2} + \frac{A(-11,-6,-14)q^{-2}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(13,8,14)q^{-2}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(5,7,6)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}} + \frac{A(3,2,4)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(-1,-6,0)q^{-3}J_0J_6^2J_8J_{11}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{12}J_{13}J_{14}^2} + \frac{A(-4,-1,-6)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{14}} + \frac{A(-1,1,0)q^{-3}J_0J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(-4,-7,-5)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{13}J_{14}^2} \\
 & + \frac{A(1,-2,1)q^{-3}J_0J_6^2J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-3,-2,-2)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} + \frac{A(3,2,2)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{12}J_{13}J_{14}} + \frac{A(3,2,2)q^{-3}J_0J_6^2J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{14}^2} \\
 & + \frac{A(0,-1,1)q^{-3}J_0J_6^2J_7J_{12}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{13}J_{14}} + \frac{A(0,1,-1)q^{-3}J_0J_6^2J_7J_{14}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}^2} + \frac{A(2,1,0)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(-2,-2,0)q^{-2}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(1,1,1)q^{-1}J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1,1,-2)q^{-3}J_0J_8J_9^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1,-1,0)q^{-3}J_0J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(-1,-1,-1)q^{-1}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1,-1,-1)J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(1,1,0)qJ_0J_5^2J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_7J_9J_{10}J_{11}J_{12}J_{13}J_{14}^2}, \\
 R2_3(q) = & \frac{A(2,2,1)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}^2} + \frac{A(-2,0,-2)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(5,0,2)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}^2} \\
 & + \frac{A(-1,-3,0)q^{-3}J_0J_6J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1,0,-2)q^{-3}J_0J_6J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(5,2,4)q^{-3}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{12}J_{13}J_{14}^2} + \frac{A(5,4,5)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(3,4,2)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}^2} + \frac{A(-9,-4,-7)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-4,-5,0)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(-3,-3,-1)q^{-3}J_0J_6J_7^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(10,7,6)q^{-3}J_0J_6J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{13}J_{14}} + \frac{A(5,3,3)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{13}J_{14}} + \frac{A(-5,-2,-4)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}} \\
 & + \frac{A(3,2,2)q^{-2}J_0J_6J_7^2J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(1,3,0)q^{-3}J_0J_6^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}} + \frac{A(0,4,0)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{12}J_{13}J_{14}} + \frac{A(9,13,4)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}} \\
 & + \frac{A(0,-4,0)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1,1,0)q^{-3}J_0J_6^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} + \frac{A(0,1,-1)q^{-3}J_0J_6^2J_9J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} + \frac{A(-7,-13,-2)q^{-3}J_0J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}^2}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{A(-12, -11, -6)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{12}J_{13}J_{14}} + \frac{A(4, 1, 2)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}} + \frac{A(-3, -2, -2)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-2, 1, -2)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{10}J_{12}J_{13}J_{14}} \\
& + \frac{A(-10, -12, -5)q^{-2}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(3, 3, 1)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(7, 11, 3)q^{-2}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(0, -1, 0)q^{-3}J_0J_7J_8^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-1, 1, 0)q^{-3}J_0J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}} + \frac{A(-2, -5, -1)q^{-3}J_0J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(1, 3, 0)q^{-3}J_0J_8^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(0, -1, 0)q^{-2}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(1, 0, 0)J_0J_6^2J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-4, -2, -2)qJ_0J_5J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}}, \\
R2_4(q) & = A(0, -1, 1)q^{-\frac{4}{7}}S(1, 7; \tau/7) + A(0, -1, 1)q^{-\frac{4}{7}}S(2, 7; \tau/7) + \frac{A(-6, 10, -17)q^{-3}J_0J_7^2J_8J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(4, -5, 11)q^{-3}J_0J_6J_8J_9J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(-2, 17, -15)q^{-3}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-5, -5, -2)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-2, -7, 2)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{12}J_{13}J_{14}} \\
& + \frac{A(-4, 8, -12)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(1, -18, 14)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(15, -15, 34)q^{-3}J_0J_6J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(1, 1, 1)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-2, 0, -3)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}} + \frac{A(0, -2, 2)q^{-3}J_0J_6J_7^2J_8J_{14}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}} \\
& + \frac{A(1, -8, 8)q^{-2}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(0, -5, 3)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}} + \frac{A(0, 5, -3)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{12}J_{13}J_{14}} + \frac{A(6, 1, 6)q^{-3}J_0J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(0, 2, -2)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(2, 1, 4)q^{-3}J_0J_6^2J_8J_{11}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{12}J_{13}J_{14}} + \frac{A(-4, -1, -5)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{12}J_{13}J_{14}} + \frac{A(-2, 3, -8)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-1, -1, -2)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(3, 2, 2)q^{-3}J_0J_6^2J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} + \frac{A(-3, 0, -4)q^{-3}J_0J_6^2J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{12}J_{13}J_{14}} + \frac{A(-3, -2, -2)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-1, 0, 1)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}} + \frac{A(5, -8, 16)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(2, 4, -1)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(1, 6, -3)q^{-3}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-2, 5, -7)q^{-2}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-9, 7, -18)q^{-3}J_0J_6^2J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(1, 5, -4)q^{-3}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(0, -2, 2)q^{-3}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}} \\
& + \frac{A(1, 3, -1)q^{-2}J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(4, 7, -1)q^{-3}J_0J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-2, -2, -1)q^{-3}J_0J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(0, -1, 1)q^{-3}J_0J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(0, -1, 1)q^{-3}J_0J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(0, -8, 8)q^{-3}J_0J_6J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(0, 4, -4)q^{-3}J_0J_6J_8^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(0, -1, 1)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(0, -1, 1)q^{-1}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(1, -1, 0)q^{-3}J_0J_6^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}} + \frac{A(0, -1, -1)J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(1, 5, -2)qJ_0J_5J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}}, \\
R2_5(q) & = \frac{A(21, 15, 5)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-13, -8, -2)q^{-3}J_0J_6J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-4, 0, -2)q^{-3}J_0J_6J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-4, -6, -1)q^{-3}J_0J_6J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(23, 13, 9)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-25, -15, -6)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-22, -17, -9)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-13, -6, -6)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-5, -2, -3)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}J_{12}J_{13}J_{14}} + \frac{A(-2, -1, -2)q^{-3}J_0J_6^2J_8J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(9, 6, 3)q^{-3}J_0J_6^2J_9J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-21, -11, -7)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(9, 8, 2)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(-8, -6, -1)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(15, 10, 5)q^{-3}J_0J_6^2J_8^2J_{11}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{12}J_{13}J_{14}} \\
& + \frac{A(11, 3, 5)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(3, 2, 2)q^{-3}J_0J_6^2J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(10, 10, 2)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}J_{12}J_{13}J_{14}} + \frac{A(-15, -14, -4)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{12}J_{13}J_{14}} \\
& + \frac{A(19, 10, 7)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(4, 1, 0)q^{-3}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{13}J_{14}} + \frac{A(0, 1, 2)q^{-3}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{12}J_{13}} \\
& + \frac{A(-7, -8, -2)q^{-2}J_0J_6^2J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(6, 4, 1)q^{-3}J_0J_6^2J_7^2J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-9, -7, -2)q^{-3}J_0J_6^2J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(2, 2, 1)q^{-3}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(9, 7, 2)q^{-3}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}} + \frac{A(6, 4, 1)q^{-3}J_0J_6^2J_7^2J_{12}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{10}J_{11}J_{13}J_{14}} + \frac{A(-6, -4, -1)q^{-3}J_0J_6^2J_7^2J_{14}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{10}J_{11}J_{12}J_{13}} \\
& + \frac{A(7, 8, 2)q^{-2}J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}} + \frac{A(-4, -4, 0)q^{-3}J_0J_8J_7J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(11, 8, 4)q^{-3}J_0J_7J_8J_{10}^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}J_{12}J_{13}J_{14}} + \frac{A(-4, -1, -2)q^{-3}J_0J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(1, 1, 0)q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}} + \frac{A(1, -1, 0)q^{-3}J_0J_6J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{11}J_{12}J_{13}} + \frac{A(-5, -2, -3)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{13}J_{14}} + \frac{A(-2, -1, -2)J_0J_5J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(5, 2, 3)qJ_0J_5J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_7J_9J_{10}J_{11}J_{12}J_{13}J_{14}}, \\
R2_6(q) & = A(1, -1, 0)q^{-\frac{6}{7}}S(4, 7; \tau/7) + A(-1, 1, 0)q^{-\frac{6}{7}}S(6, 7; \tau/7) + \frac{A(31, -12, -5)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}J_{14}} \\
& + \frac{A(-11, 5, 1)q^{-3}J_0J_6J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}J_{11}J_{12}J_{13}J_{14}} + \frac{A(3, 6, 1)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(-17, 4, 2)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}J_{12}J_{13}J_{14}} + \frac{A(15, -6, -1)q^{-3}J_0J_6J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}J_{11}J_{12}J_{13}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{A(-9,6,5)q^{-3}J_0J_6J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-3,-6,-4)q^{-3}J_0J_6^2J_8^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-7,8,5)q^{-3}J_0J_6^2J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-2,2,3)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-7,2,0)q^{-3}J_0J_6^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(9,0,-1)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(4,2,1)q^{-3}J_0J_6^2J_7J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-8,-8,-7)q^{-3}J_0J_6^2J_7}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,-2,0)q^{-3}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-13,10,7)q^{-3}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(10,-3,-3)q^{-3}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-2,2,1)q^{-3}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(1,-1,0)q^{-3}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(3,2,2)q^{-3}J_0J_6^2J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-3,-6,-4)q^{-3}J_0J_6^2J_7^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-1,2,0)q^{-3}J_0J_6^2J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(3,-3,-1)q^{-3}J_0J_6^2J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-4,1,-1)q^{-3}J_0J_6^2J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-14,5,4)q^{-3}J_0J_6^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(0,1,-1)q^{-3}J_0J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(24,-2,3)q^{-3}J_0J_7J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-12,-1,-4)q^{-3}J_0J_7J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-24,2,-3)q^{-3}J_0J_7J_8^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(8,1,2)q^{-3}J_0J_7^2J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-7,2,-2)q^{-3}J_0J_6J_8^2J_9^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-4,-4,0)q^{-3}J_0J_6J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(10,4,6)q^{-3}J_0J_6J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-3,-2,-4)q^{-3}J_0J_6J_8J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(33,-7,0)q^{-3}J_0J_6J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(0,-1,0)q^{-3}J_0J_6J_7J_8J_{14}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-3,-3,-1)q^{-3}J_0J_6J_7^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-1,1,0)q^{-3}J_0J_6J_7^2J_{12}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(7,-2,2)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(-7,2,-2)q^{-3}J_0J_6^2J_8J_9}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-3,-2,-2)q^{-3}J_0J_6^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_7J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(-1,1,0)q^{-3}J_0J_6^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_8J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(2,-2,0)q^{-1}J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(0,0,-1)J_0J_6^2J_7J_8}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(4,0,1)q^{-3}J_0J_7J_8^2J_{10}}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} + \frac{A(0,-1,0)q^{-3}J_0J_7^2J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_6J_9J_{11}^2J_{12}^2J_{13}^2J_{14}^2} \\
& + \frac{A(0,1,0)q^{-3}J_0J_8^2}{J_1^2J_2^2J_3^2J_4^2J_5^2J_{10}^2J_{11}^2J_{12}^2J_{13}^2J_{14}^2}.
\end{aligned}$$

Due to the length of the identities, we perform the calculations in Maple and only sketch the proof here. We note that while the calculations are quite long in terms of their statements, they are quite quick to compute. A Maple file containing calculations relevant to proving Theorem 2.2 can be found in the publications section of the author's website, <http://people.oregonstate.edu/~jennichr/#research>.

We multiply both sides of the identity in Theorem 2.2 by  $q^{-\frac{1}{8}}$ , in doing so we find that the exponents simplify correctly to give an identity between  $\mathcal{R}2(1, 7; \tau)$ , various  $\mathcal{S}(k, 8; \tau)$ , and various quotients of  $f_{196, 7k}(\tau)$ . When then multiply both sides by  $\frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)}$  and take the difference of both sides, we call this difference  $LHS - RHS$ . We are to show  $0 = LHS - RHS$ . Upon inspection we find that the resulting generalized eta quotients are all modular forms of weight 1 on  $\Gamma_0(196) \cap \Gamma_1(28)$  by Corollary 4.17 and more importantly the roots of unity have been chosen correctly so that part (3) of Theorem 2.1 applies to give that the remaining terms also constitute a weight 1 modular form on  $\Gamma_0(196) \cap \Gamma_1(28)$ . We then choose one of the generalized eta quotients, call it  $g_1$ , so that  $\frac{LHS - RHS}{g_1}$  is a modular function on  $\Gamma_0(196) \cap \Gamma_1(28)$ . We then verify that  $0 = \frac{LHS - RHS}{g_1}$  according to the valence formula.

To state the valence formula, we need to recall the concept of the order at a point with respect to a group. Suppose  $f$  is a modular function on some congruence subgroup  $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ . Suppose  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , we then have a cusp  $\zeta = A(\infty) = \frac{\alpha}{\gamma}$ . We let  $\mathrm{ord}(f; \zeta)$  denote the invariant order of  $f$  at  $\zeta$ . We define the width of  $\zeta$  with respect to  $\Gamma$  as  $\mathrm{width}_\Gamma(\zeta) := w$ , where  $w$  is the least positive integer such that  $A \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} A^{-1} \in \Gamma$ . We then define the order of  $f$  at  $\zeta$  with respect to  $\Gamma$  as  $\mathrm{ORD}_\Gamma(f; \zeta) = \mathrm{ord}(f; \zeta) \mathrm{width}_\Gamma(\zeta)$ . For  $z \in \mathcal{H}$  we let  $\mathrm{ord}(f; z)$  denote the order of  $f$  at  $z$  as a meromorphic function. We then define the order of  $f$  at  $z$  with respect to  $\Gamma$  as  $\mathrm{ORD}_\Gamma(f; z) = \mathrm{ord}(f; z)/m$  where  $m$  is the order of  $z$  as a fixed point of  $\Gamma$  (so  $m = 1, 2$ , or  $3$ ).

The valence formula, for modular functions can be stated as follows. Suppose  $f$  is a modular function that is not the zero function and  $\mathcal{D} \subset \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$  is a fundamental domain for the action of  $\Gamma$  on  $\mathcal{H}$  along with a complete set of inequivalent cusps for the action, then

$$\sum_{\zeta \in \mathcal{D}} \mathrm{ORD}_\Gamma(f; \zeta) = 0.$$

A complete set of inequivalent cusps, along with their widths, for  $\Gamma_0(196) \cap \Gamma_1(28)$  is given in Table 1. We let  $\mathcal{D}$  denote these cusps along with a fundamental region of the action of  $\Gamma$ .

We note  $\frac{LHS - RHS}{g_1}$  has no poles in  $\mathcal{H}$ , but it may have zeros in  $\mathcal{H}$ . We take a lower bound on the orders at the non-infinite cusps by taking the minimum order of each of the individual summands, which we compute

TABLE 1.

cuspidal	0	$\frac{1}{28}$	$\frac{3}{80}$	$\frac{2}{53}$	$\frac{1}{26}$	$\frac{3}{77}$	$\frac{2}{51}$	$\frac{1}{24}$	$\frac{2}{47}$	$\frac{3}{70}$	$\frac{2}{45}$	$\frac{5}{112}$	$\frac{1}{22}$	$\frac{1}{21}$	$\frac{1}{20}$	$\frac{5}{98}$	$\frac{3}{56}$	$\frac{1}{18}$	$\frac{2}{35}$	$\frac{1}{16}$	$\frac{1}{15}$	$\frac{1}{14}$
width	196	1	49	196	98	4	196	49	196	2	196	1	98	4	49	2	1	98	4	49	196	2
cuspidal	$\frac{5}{63}$	$\frac{1}{12}$	$\frac{3}{35}$	$\frac{8}{91}$	$\frac{5}{56}$	$\frac{2}{21}$	$\frac{11}{112}$	$\frac{5}{49}$	$\frac{8}{77}$	$\frac{3}{28}$	$\frac{5}{42}$	$\frac{11}{91}$	$\frac{6}{49}$	$\frac{1}{8}$	$\frac{8}{63}$	$\frac{9}{70}$	$\frac{12}{91}$	$\frac{1}{7}$	$\frac{12}{77}$	$\frac{10}{63}$	$\frac{9}{56}$	$\frac{6}{35}$
width	4	49	4	4	1	4	1	4	4	1	2	4	4	49	4	2	4	4	4	4	1	4
cuspidal	$\frac{11}{63}$	$\frac{16}{91}$	$\frac{5}{28}$	$\frac{13}{70}$	$\frac{4}{21}$	$\frac{10}{49}$	$\frac{13}{63}$	$\frac{22}{105}$	$\frac{3}{14}$	$\frac{11}{49}$	$\frac{19}{84}$	$\frac{13}{56}$	$\frac{18}{77}$	$\frac{5}{21}$	$\frac{20}{77}$	$\frac{51}{196}$	$\frac{15}{56}$	$\frac{2}{7}$	$\frac{25}{84}$	$\frac{23}{77}$	$\frac{17}{56}$	$\frac{13}{42}$
width	4	4	1	2	4	4	4	4	2	4	1	1	4	4	4	1	1	4	1	4	1	2
cuspidal	$\frac{11}{35}$	$\frac{9}{28}$	$\frac{19}{56}$	$\frac{31}{91}$	$\frac{67}{196}$	$\frac{12}{35}$	$\frac{29}{84}$	$\frac{69}{196}$	$\frac{5}{14}$	$\frac{75}{196}$	$\frac{11}{28}$	$\frac{17}{42}$	$\frac{23}{56}$	$\frac{81}{196}$	$\frac{29}{70}$	$\frac{3}{7}$	$\frac{43}{98}$	$\frac{25}{56}$	$\frac{45}{98}$	$\frac{13}{28}$	$\frac{10}{21}$	$\frac{27}{56}$
width	4	1	1	4	1	4	1	1	2	1	1	1	2	1	2	4	2	1	2	1	4	1
cuspidal	$\frac{41}{84}$	$\frac{43}{84}$	$\frac{29}{56}$	$\frac{15}{28}$	$\frac{23}{42}$	$\frac{4}{7}$	$\frac{37}{63}$	$\frac{29}{49}$	$\frac{25}{42}$	$\frac{17}{28}$	$\frac{13}{21}$	$\frac{22}{35}$	$\frac{9}{14}$	$\frac{55}{84}$	$\frac{19}{28}$	$\frac{29}{42}$	$\frac{39}{56}$	$\frac{59}{84}$	$\frac{5}{7}$	$\frac{37}{49}$	$\frac{53}{70}$	$\frac{65}{84}$
width	1	1	1	1	2	4	4	4	2	1	4	4	2	1	1	2	1	1	4	4	2	1
cuspidal	$\frac{11}{14}$	$\frac{23}{28}$	$\frac{6}{7}$	$\frac{61}{70}$	$\frac{25}{28}$	$\frac{101}{112}$	$\frac{13}{14}$	$\frac{107}{112}$	$\frac{27}{28}$	$\infty$												
width	2	1	4	2	1	1	2	1	1	1												

with Propositions 5.2, 5.3, and 5.4. This lower bound yields

$$\sum_{\zeta \in \mathcal{D}} \text{ORD}_{\Gamma} \left( \frac{LHS - RHS}{g_1}; \zeta \right) \geq \text{ord} \left( \frac{LHS - RHS}{g_1}, \infty \right) - 456.$$

However, we can expand  $\frac{LHS - RHS}{g_1}$  as a series in  $q$  and find the coefficients of  $\frac{LHS - RHS}{g_1}$  are zero to at least  $q^{457}$ . Thus

$$\sum_{\zeta \in \mathcal{D}} \text{ORD}_{\Gamma} \left( \frac{LHS - RHS}{g_1}; \zeta \right) \geq 1,$$

and so  $\frac{LHS - RHS}{g_1}$  must be identically zero by the valence formula. This establishes Theorem 2.2.

## 7. REMARKS

As we have currently constructed our functions, the choice of  $\tilde{\mathcal{S}}(k, c; \tau)$  is best possible when  $c$  is odd in the sense that it agrees with the multiplier for  $\tilde{\mathcal{R}}_2(a, c; \tau)$  on the subgroup given in Corollary 4.4. When  $c$  is even, the subgroup in Corollary 4.4 simplifies to  $\Gamma_0(2c^2) \cap \Gamma_1(2c)$ , however  $\tilde{\mathcal{S}}(k, c; \tau)$  requires us to work on  $\Gamma_0(8c^2) \cap \Gamma_1(4c)$ . It would be interesting to see if it is possible to improve the definition of  $\mathcal{S}(k, c; \tau)$  when  $c$  is even and further improve the subgroup when  $c$  is odd. We have emphasized choosing  $\tilde{\mathcal{S}}(k, c; \tau)$  to determine the  $c$ -dissection of  $R_2(a, c; \tau)$ , but different functions would also be of use for other types of identities.

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